The Composition operator induced by a polynomial of degree n

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Abstract:

In this paper, we characterize normal composition operators induced by holomorphic self-map $\varphi(z) = az^2 + bz + c$, when $|a| \le 1, |b| \le 1, |c| < 1$ and $|a| \le 1, |b| \le 1, |c| < 1$ .Moreover, we study other related classes of operators, and then we generalize these results to polynomials of degree n.

Key words: Composition operator: Normality : Unitary operator

Introduction:

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the collection of holomorphic (analytic)functions $f(z) = \sum_{n=o}^{\infty} \hat{f}(n) z^n$ with $\hat{f}(n)$ denoting the n-th Taylor coefficient of f such that $\sum_{n=o}^{\infty} \left| \hat{f}(n) \right|^2 < \infty$

and the norm is defined by $// f //^2 = \sum_{n=o}^{\infty} \left| \hat{f}(n) \right|^2$.

The particular importance of H^2 is due to the fact that it is a Hilbert space with inner product on H^2 is defined by $< f, g >= \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)}$,where $\hat{f}(n)$ and $\hat{g}(n)$ are *n*-th Taylor coefficient of f and g in H^2 . Let ψ be a holomorphic function that take the unit ball U into itself (which is called holomorphic self-map of U).To each holomorphic self-map ψ of U, we associate the composition operator C_ψ defined for all $f \in H^2$ by

$C_\psi f = f \circ \psi$.

In this paper, we are going to discuss some links between the function theory and the operator theory .We investigate the relationship between the properties of symbol φ and the operator C_φ where $\varphi(z) = az^2 + bz + c$ such that $|a| \le 1, |b| \le 1, |c| < 1$ and $|a| \le 1, |b| \le 1, |c| < 1$. Composition operators have been studied by many authors in different contexts. A good source of references on the properties of composition operators on H^2 can be found in [1]. We state very loosely some basic facts on composition operators on H^2.

Theorem 1: Every composition operator C_ψ is bounded.

Theorem 2: C_ψ is normal if and only if $\psi(z) = \lambda z$, $|\lambda| \le 1$.

Theorem 3: $C_\delta C_\psi = C_{\psi \circ \delta}$.

Theorem 4: C_ψ is an identity operator if and only if ψ is the identity map.

For each $\alpha \in U$, the reproducing kernel at α,denoting by K_α is defined by

$$K_\alpha(z) = \frac{1}{1 - \overline{\alpha} z}.$$

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It is easily seen for each $\alpha \in U$ and $f \in H^2$, $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ that $< f, K_\alpha >= \sum_{n=0}^{\infty}\hat{f}(n)\alpha^n = f(\alpha)$,[2].

The reproducing kernels for H^2 will play an important role in the study of composition operators since the span of this family $\{K_\alpha\}_{\alpha\in U}$ is a dense subset in H^2. Shapiro in [1] gave the following formula for the adjiont C_ψ^* of a composition operator C_ψ induced by a holomorphic self-map ψ of U on that family.

Theorem 5 : Let ψ be a holomorphic self-map of U, then for each $\alpha \in U$ $C_\psi^* K_\alpha = K_{\psi(\alpha)}$.

Cowen in [3] gave an exact value of composition operator induced by polynomial of degree 1.

Theorem 6: Let $\psi(z) = sz + t$ where $/s/\le 1, /t/<1$ and $/s/+/t/\le 1$,the norm of C_ψ on H^2 is defined as follows

$$//C_\psi//^2 = \frac{2}{1-|s|^2-|t|^2+\sqrt{(1-|s|^2+|t|^2)^2-4|t|^2}}.$$

In this paper, we study the normality of a composition operator induced by a holomorphic self-map $\varphi(z)=az^2+bz+c$ where $|a|\le 1,|b|\le 1,|c|<1$ and $|a|\le 1,|b|\le 1,|c|\le 1$. In addition we study other related classes, and extend these results to a polynomial of degree n.

1. The characterization of the normality of C_φ.

Recall that [4] an operator T on a Hilbert space H is said to be normal if $TT^* = T^*T$ (where T^* is the adjoint of T) and is isometric if $T^*T = I$ (where I is the identity operator). Moreover, T is unitary if $TT^* = T^*T = I$ .We start this section by the following consequence.

Theorem 1.1:
Let $\varphi(z)=az^2+bz+c$, where $|a|\le 1,|b|\le 1,|c|<1$ and $|a|\le 1,|b|\le 1,|c|\le 1$. If $|b|=1$,then C_φ is an isometric on H^2.

Proof :

Assume that $|b|=1$, so it is clear that a=c=0 (since $|a|\le 1,|b|\le 1,|c|\le 1$).Therefore $\varphi(z)=bz$.To prove that C_φ is isometric it is enough to show that $C_\varphi^* C_\varphi = I$.Since the span of the family $\{K_\alpha\}_{\alpha\in U}$ is dense in H^2, then we can prove the equality on this family. Let $\alpha \in U$, then by theorem (5)

$$C_\varphi^* C_\varphi K_\alpha(z) = C_\varphi^* K_\alpha(\varphi(z))$$

$= K_{\varphi(\alpha)}(\varphi(z))$ (by theorem 5)

$$= \frac{1}{1-\overline{\varphi(\alpha)}(\varphi(z))}$$

$$= \frac{1}{1-\bar{b}\,\bar{\alpha} bz}$$

$$= \frac{1}{1-|b|^2\bar{\alpha}z}$$

$$= \frac{1}{1-\bar{\alpha}z}$$

$= K_\alpha(z)$.

Hence $C_\varphi^* C_\varphi K_\alpha(z) = K_\alpha(z)$ for each $\alpha \in U$ .This implies that $C_\varphi^* C_\varphi = I$ .So C_φ is isometric.

By using similar technique of (1.1) we can generalize this theorem to a polynomial of degree n.

Theorem 1.2:
Let $\varphi_n(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0$ where $|a_i|\le 1$, i=1,2,…,n , $|a_0|<1$ and $\sum_{i=0}^{n}|a_i|\le 1$. If $|a_1|=1$, then C_{φ_n} is isometric operator on H^2.

The following result gives the necessary and sufficient condition for normality of C_φ.

Theorem 1.3:
Let $\varphi(z)=az^2+bz+c$ where and $|a|\le 1,|b|\le 1,|c|<1,$
$|a|\le 1,|b|\le 1,|c|\le 1$ .Then C_φ is normal if and only if a=c=0.

Proof :
Assume that C_φ is normal. Trivial case when C_φ is the identity operator, then by theorem (4) φ is the identity self-map, hence $\varphi(z)=z$, thus a=c=0. Therefore we may assume that C_φ is not the identity operator, then φ is not identity self-map of U. To prove a=c=0, we suppose the converse, first assume that $c\ne 0$ then $\varphi(0)=c\ne 0$.But C_φ is normal, then

$C_\varphi C_\varphi^* K_0(z)=C_\varphi^* C_\varphi K_0(z)$

[Since $C_\varphi K_0=K_0$ and by theorem (5) $C_\varphi^* K_0(z)=K_{\varphi(0)}(z)$]

$C_\varphi K_{\varphi(0)}(z)=C_\varphi^* K_0(z)$

Hence $K_{\varphi(0)}(\varphi(z))=K_{\varphi(0)}(z)$. This implies that $\frac{1}{1-\overline{\varphi(0)}\varphi(z)}=\frac{1}{1-\overline{\varphi(0)}z}$.

Thus $\overline{\varphi(0)}\ \varphi(z)=\overline{\varphi(0)}\ z$. Since $\varphi(0)\ne 0$,then $\varphi(z)=z$,which is a contradiction (since φ is not identity map).Thus c=0, it follows that $\varphi(z)=az^2+bz$.Therefore $\varphi(0)=0$.This implies that z^nH^2 is an invariant subspace of H^2 under C_φ for each positive integer n (by [5]). But C_φ is normal, then by [6] $(z^nH^2)^\perp$ is also invariant under C_φ.In particular, $(z^2H^2)^\perp$ is an invariant subspace of H^2 under C_φ. But $(z^2H^2)^\perp=span\{1,z\}$, then $C_\varphi z\in span\{1,z\}$ .It follows that $C_\varphi z=\varphi(z)=\alpha+\beta z$ for some α,β .But $\varphi(z)=az^2+bz$,therefore $\alpha=0, a=0$ and $\beta=b$,as desired.

Conversely, if a=c=0, then $\varphi(z)=bz$ with $|a|\le 1$, then again by theorem (2) C_φ is normal.

The next consequence is a generalization of (1.3).

Corollary 1.4:
Let $\varphi_n(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0$ where $|a_0|<1$, $|a_i|\le 1$,i=1,2,…,n and $\sum_{i=0}^{n}|a_i|\le 1$.Then C_{φ_n} is normal if and only if $a_i=0$,i=0,…,n , $i\ne 1$.

Corollary 1.5:
C_φ is a unitary operator on H^2 if and only if /b/=1.

Proof :
Assume that C_φ is unitary, then $TT^*=T^*T=I$.
Since C_φ is unitary then it is normal, thus by theorem (1.3) a=c=0, it follows that $\varphi(z)=bz$, $|b|\le 1$. To show that $|b|=1$. Let $\alpha\in U$, $\alpha\ne 0$, then

$$C_\varphi^* C_\varphi K_\alpha(z)=C_\varphi^* K_\alpha(\varphi(z))$$
$$=K_{\varphi(\alpha)}(\varphi(z))$$
$$=\frac{1}{1-\overline{\varphi(\alpha)}\varphi(z)}.$$

But
$$C_\varphi^* C_\varphi K_\alpha(z)=I(K_\alpha(z))=K_\alpha(z).$$

Thus $\frac{1}{1-\overline{\varphi(\alpha)}\varphi(z)} = \frac{1}{1-\overline{\alpha}z}$.Therefore, $\frac{1}{1-\overline{b}\,\overline{\alpha}bz} = \frac{1}{1-\overline{\alpha}z}$ .Hence $\frac{1}{1-|b|^2\overline{\alpha}z} = \frac{1}{1-\overline{\alpha}z}$. This equation satisfies only if $|b|=1$.

Conversely, suppose that $|b|=1$. But $|a|\le 1, |b|\le 1, |c|\le 1$, this follows that a=c=0. Hence by (1.3) we obtain C_φ is normal, that is $C_\varphi C_\varphi^* = C_\varphi^* C_\varphi$.On the other hand, since $|b|=1$, by (1.1) C_φ is isometric. Thus $C_\varphi C_\varphi^* = C_\varphi^* C_\varphi = I$ .Hence C_φ is a unitary operator on H^2.

The following result can get directly from generalize (1.5).

<u>Corollary 1.6</u>:
Let $\varphi_n(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ where $|a_0| < 1$, $|a_i| \le 1$,i=1,2,…,n and $\sum_{i=0}^{n}|a_i| \le 1$.Then C_{φ_n} is a unitary operator if and only if $|a_1| = 1$.

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المؤثر التركيبي المحتث من متعددة حدود من الدرجة n

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الخلاصة:

في هذا البحث أعطينا وصفاً للمؤثرات الاعتيادية المحتثة من الدالة التحليلية على U، $\varphi(z) = az^2 + bz + c$ عندما $|a|\le 1, |b|\le 1, |c| < 1$ و $|a|\le 1, |b|\le 1, |c| < 1$. فضلاً عن ذلك فقد درسنا بعض الأنواع الأخرى من المؤثرات، ثم قمنا بتعميم تلك النتائج على متعددة حدود من الدرجة n.