Numerical Solution of Linear integro-ordinary Differential Equations Using Taylor Series

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Abstract

Taylor series is applied to treat two types of higher order linear integro-ordinary differential equations: linear Fredholm and Volterra integro-ordinary differential. In this technique Taylor series is substituted for the unknown function after differentiating both sides of linear integro-ordinary differential equation with respect to x. The required derivatives and the involved integrals have been approximated using central differences and quadrature rules respectively. The resulting algebraic equations are solved by Gauss elimination technique. Program is written in MATLAB language and examples are presented to illustrate the results of this method.

1.Introduction

Methods for the numerical solution of linear integro-ordinary differential equations have been proposed by many investigators [3,4] , integral equation was solved using Taylor series [1].In this paper two types of linear integro-ordinary differential equations: linear Fredholm and Volterra integro-ordinary differential equations of order n are solved with the aid of Taylor series. In this research the involved derivatives were computed numerically using central differences [2]. Also the contained integrals were evaluated numerically using Romberg rule [2]. Moreover Gauss elimination technique was used to solve the system of equations. Examples with satisfactory results are given.

2.Linear Fredholm integro-ordinary differential equations

Let us consider linear Fredholm integro-ordinary differential equations of order n:

$$[D^{(n)} + \sum_{m=0}^{n-1} P_m(x) D^{(m)}]U(x) = \int_a^b k(x,t)U(t)dt \,, \; x \in I = [a,b]. \qquad (1)$$

With two-point boundary conditions:

$$\left.\begin{array}{l} U(a) = a_0, U'(a) = a_1, \dots, U^{(n-1)}(a) = a_{n-1}, \\ U(b) = b_0, U'(b) = b_1, \dots, U^{(n-1)}(b) = b_{n-1}. \end{array}\right\} \qquad (2)$$

For $U \in C^{(n)}[a,b]$ and $D^{(i)}U$ denote the *ith* derivative of U, for $i = 0,1,\dots,n$. The functions f and $P_i, i = 0,1,\dots,n-1$ are assumed to be continuous on I and k:S$\rightarrow$R (with: $S = \{(x,t) : a \le t \le x \le b\}$) denote a given continuity function. While a and b are constants.

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In order to solve (1)-(2) we follow these steps:

(I) Differentiate (1) twice with respect to x and then put x=a, to get:

$$U^{(n)}(a) = f(a) - \sum_{m=0}^{n-1} P_m(a)U^{(m)}(a) + \int_a^b k(x,t)\Big|_{x=a} U(t)dt,$$

$$U^{(n+1)}(a) = f'(a) - \sum_{m=0}^{n-1}\begin{pmatrix} P'_m(a)U^{(m)}(a) + \\ P_m U^{(m+1)}(a)\end{pmatrix} + \int_a^b \frac{\partial k(x,t)}{\partial x}\Big|_{x=a} U(t)dt,$$

$$U^{(n+2)}(a) = f''(a) - \sum_{m=0}^{n-1}\begin{pmatrix} P''_m(a)U^{(m)}(a) + \\ 2P'_m(a)U^{(m+1)}(a) + \\ P_m(a)U^{(m+2)}(a)\end{pmatrix} + \int_a^b \frac{\partial^2 k(x,t)}{\partial x^2}\Big|_{x=a} U(t)dt$$

(3)

(II) Expand U(t) in Taylor series at t=a ,that is:

$$U(t) = \sum_{k=0}^{\infty} \frac{(t-a)^k}{k!} U^{(k)}(a)1 \qquad (4)$$

(III) Put (4) in (3) this yields:

$$U^{(n)}(a) = f(a) - \sum_{m=0}^{n-1} P_m(a)U^{(m)}(a) + \int_a^b k(x,t)\Big|_{x=a}\left[\sum_{k=0}^{\infty}\frac{(t-a)^k}{k!}U^{(k)}(a)\right]dt,$$

$$P_{n-1}(a)U^{(n)}(a)+U^{(n+1)}(a)= f'(a)-\sum_{m=0}^{n-1} P'_m(a)U^{(m)}(a)-\sum_{m=0}^{n-2} P_m(a)U^{(m+1)}(a)+ \int_a^b \frac{\partial k(x,t)}{\partial x}\Big|_{x=a}\left[\sum_{k=0}^{\infty}\frac{(t-a)^k}{k!}U^{(k)}(a)\right]dt,$$

$$[2P'_{n-1}(a)+P_{n-2}(a)]U^{(n)}(a)+ P_{n-1}(a)U^{(n+1)}(a)+U^{(n+2)}(a)=f''(a)- \sum_{m=0}^{n-1} P''_m(a)U^{(m)}(a)- 2\sum_{m=0}^{n-2} P'_m(a)U^{(m+1)}(a) - \sum_{m=0}^{n-3} P_m(a)U^{(m+2)}(a)+ \int_a^b \frac{\partial^2 k(x,t)}{\partial x^2}\Big|_{x=a}\left[\sum_{k=0}^{\infty}\frac{(t-a)^k}{k!}U^{(k)}(a)\right]dt.$$

Rewrite the equations above as:

$$U^{(n)}(a) = f(a) - \sum_{m=0}^{n-1} P_m(a)U^{(m)}(a) + \sum_{k=0}^{\infty} T_{0k}U^{(k)}(a),$$

$$P_{n-1}(a)U^{(n)}(a) + U^{(n+1)}(a) = f'(a) - \sum_{m=0}^{n-1} P'_m(a)U^{(m)}(a) - \sum_{m=0}^{n-2} P_m(a)U^{(m+1)}(a) + \sum_{k=0}^{\infty} T_{1k}U^{(k)}(a),$$

$$[2P'_{n-1}(a) + P_{n-2}(a)]U^{(n)}(a) + P_{n-1}(a)U^{(n+1)}(a) + U^{(n+2)} = f''(a) - \sum_{m=0}^{n-1} P''_m(a)U^{(m)}(a) - 2\sum_{m=0}^{n-2} P'_m(a)U^{(m+1)}(a) - \sum_{m=0}^{n-3} P_m(a)U^{(m+2)}(a) + \sum_{k=0}^{\infty} T_{2k}U^{(k)}(a)$$

Where

$$T_{ik} = \frac{1}{(n+k)!}\int_a^b \frac{\partial^i k(x,t)}{\partial x^i}\Big|_{x=a}(t-a)^{n+k}\,dt\ , i = 0,1,2.$$

(IV) After solving the following system, the values $U^{(n)}(a), U^{(n+1)}(a) and U^{(n+2)}(a)$ are obtained:

$$\left.\begin{array}{l}
[1-T_{00}]U^{(n)}(a)-T_{01}U^{(n+1)}(a)- \\
T_{02}U^{(n+2)}(a)=f(a)- \\
p_m(a)U^{(m)}(a)+ \\
\sum_{m=0}^{n-1}\int_a^b k(x,t)\Big|_{x=a}\left(\sum_{j=0}^{n-1}\frac{(t-a)^j}{j!}U^{(j)}(a)\right)dt \\
{[P_{n-1}(a)-T_{10}]U^{(n)}(a)+} \\
{[1-T_{11}]U^{(n+1)}(a)-T_{12}U^{(n+2)}(a)=} \\
f'(a)-\sum_{m=0}^{n-1}P'_m(a)U^{(m)}(a)-\sum_{m=0}^{n-2}P_m(a)U^{(m+1)}(a) \\
+\int_a^b\frac{\partial k(x,t)}{\partial x}\Big|_{x=a}\left(\sum_{j=0}^{n-1}\frac{(t-a)^j}{j!}U^{(j)}(a)\right)dt \\
{[P_{n-2}(a)+2P'_{n-1}(a)-T_{20}]U^{(n)}(a)+} \\
{[P_{n-1}(a)-T_{21}]U^{(n+1)}(a)+[1-T_{22}]U^{(n+2)}(a)=} \\
f''(a)-\sum_{m=0}^{n-1}P''_m(a)U^{(m)}(a) \\
\qquad P'_m(a)U^{(m+1)}(a)-\sum_{m=0}^{n-3}P_m(a)U^{(m+2)}(a)+ \\
-2\sum_{m=0}^{n-2}\int_a^b\frac{\partial^2 k(x,t)}{\partial x^2}\Big|_{x=a}\left(\sum_{j=0}^{n-1}\frac{(t-a)^j}{j!}U^{(j)}(a)\right)dt
\end{array}\right\}$$

(5)

(V) Finally using step (IV) and the values $U(a), U'(a), \ldots, U^{(n-1)}(a)$ from (2),we obtain $U^{(n)}(a), U^{(n+1)}(a) and U^{(n+2)}(a)$.

Algorithm (1)

To find the approximate solution of (1)-(2), first select positive integers $i \; and \; k$, let $i=k=2$, choose a suitable h and perform the following steps:

Step (1) To evaluate $f'(a) and f''(a)$ set

$$f'(a)=\frac{-f(a+2h)+8f(a+h)-8f(a-h)+f(a-2h)}{12h}$$

$$f''(a)=\frac{f(a+h)-2f(a)+f(a-h)}{h^2}$$

Step (2) To calculate $P'_i(a) and P''_i(a) for \; i=1,2,\ldots,n-1$ set

$$P'_i(a)=\frac{-P_i(a+2h)+8P_i(a+h)-8P_i(a-h)+P_i(a-2h)}{12h}$$

$$P''_i(a)=\frac{P_i(a+h)-2P_i(a)+P_i(a-h)}{h^2}$$

Step (3) To calculate $\frac{\partial k(x,t)}{\partial x}\Big|_{x=a} and \frac{\partial^2 k(x,t)}{\partial x^2}\Big|_{x=a}$ set

$$\frac{\partial k(x,t)}{\partial x}\Big|_{x=a}=\frac{-k(a+2h,t)+8k(a+h,t)-8k(a-h,t)+k(a-2h,t)}{12h}$$

$$\frac{\partial^2 k(x,t)}{\partial x^2}\Big|_{x=a}=\frac{k(a+h,t)-2k(a,t)+k(a-h,t)}{h^2}$$

Step (4) Approximate T_{ik} using Romberg rule.

Step (5) Solve (5) using Gauss elimination echnique o et $U^{(n)}(a), U^{(n+1)}(a) and U^{(n+2)}(a)$.

Step (6) Finally, substitute the values of $U(a), U'(a), \ldots, U^{(n+2)}(a)$ in (4) to get the approximate solution of (1)-(2), that is:

$$U(x)=U(a)+(x-a)U'(a)+\frac{(x-a)^2}{2!}U''(a)+\cdots+\frac{(x-a)^{n+2}}{(n+2)!}U^{(n+2)}(a).$$

3. Linear Volterra integro-ordinary differential Equations

Consider linear Volterra integro-ordinary differential equation of order n:

$$[D^{(n)} + \sum_{m=0}^{n-1} P_m(x) D^{(m)}]U(x) = f(x) + \int_a^x k(x,t)U(t)dt \,, x \in [a,b] \qquad (6)$$

With initial conditions:

$$U(a) = u_0, U'(a) = u_1, \dots, U^{(n-1)}(a) = u_{n-1}. \qquad (7)$$

Where the kernel k(x,t) and the function f(x) are known, U(x) has to be evaluated and $D^{(i)} = \frac{d^i}{dx^i}, for\, i = 0,1,\dots,n$.

As before after differentiating (6) two times with respect to x, we get:

$$U^{(n)}(x) = f(x) - \sum_{m=0}^{n-1} P_m(x)U^{(m)}(x) + \int_a^x k(x,t)U(t)dt,$$

$$U^{(n+1)}(x) = f'(x) - \sum_{m=0}^{n-1} [P'_m(x)U^{(m)}(x) + P_m(x)U^{(m+1)}(x)] + \int_a^x \frac{\partial k(x,t)}{\partial x} U(t)dt + k(x,x)U(x),$$

$$U^{(n+2)}(x) = f''(x) - \sum_{m=0}^{n-1} [P''_m(x)U^{(m)}(x) + 2P'_m(x)U^{(m+1)}(x) + P_m(x)U^{(m+2)}(x)] + \int_a^x \frac{\partial^2 k(x,t)}{\partial x^2} U(t)dt + \frac{\partial k(x,t)}{\partial x}\Big|_{t=x} U(x) + k'(x,x)U(x) + k(x,x)U'(x).$$

Expand U(t) in Taylor series and putting x=a, we obtain:

$$\left.\begin{aligned} U^{(n)}(a) &= f(a) - \sum_{m=0}^{n-1} P_m(a)U^{(m)}(a), \\ U^{(n+1)}(a) &= f'(a) - \sum_{m=0}^{n-1} [P'_m(a)U^{(m)}(a) + P_m(a)U^{(m+1)}(a)] + k(a,a)U(a), \\ U^{(n+2)}(a) &= f''(a) - \sum_{m=0}^{n-1} [P''_m(a)U^{(m)}(a) + 2P'_m(a)U^{(m+1)}(a) + + P_m(a)U^{(m+2)}(a)] \\ &\quad \frac{\partial k(x,t)}{\partial x}\Big|_{t=x=a} U(a) + k'(a,a)U(a) + k(a,a)U'(a). \end{aligned}\right\}$$

(8)

Solve this system and get the values of $U^{(n)}(a), U^{(n+1)}(a) and U^{(n+2)}(a)$,then substitute these values $U(a), U'(a), \dots, U^{(n+2)}(a)$ in (4) to obtain the solution U(x).

Algorithm (2)

First, choose value to h and set x=a , then perform these steps:

Step(1) Toevaluate $f'(x)\, and\, f''(x)$ set

$$f'(a) = \frac{-f(a+2h) + 8f(a+h) - 8f(a-h) + f(a-2h)}{12h}$$

$$f''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Step(2) Tocalculate $P'_i(a) and\, P''_i(a), for\; i = 1,2,\dots,n-1$ set

$$P'_i(a) = \frac{-P_i(a+2h) + 8P_i(a+h) - 8P_i(a-h) + P_i(a-2h)}{12h}$$

$$P''_i(a) = \frac{P_i(a+h) - 2P_i(a) + P_i(a-h)}{h^2}$$

Step(3) Tocalculate

$\frac{\partial k(x,t)}{\partial x}|_{x=a}$ *and* $\frac{\partial^2 k(x,t)}{\partial x^2}|_{x=a}$ set

$$\frac{\partial k(x,t)}{\partial x}|_{x=a} = \frac{-k(a+2h,t)+8k(a+h,t)-8k(a-h,t)+k(a-2h,t)}{12h}$$

$$\frac{\partial^2 k(x,t)}{\partial x^2}|_{x=a} = \frac{k(a+h,t)-2k(a,t)+k(a-h,t)}{h^2}$$

Step (4) Solve (8) using Gauss elimination echnique o et $U^{(n)}(a), U^{(n+1)}(a) and U^{(n+2)}(a)$.

Step (5) Finally, as in algorithm (1) substitute the values of in (4) to get the approximate solution of (6)-(7), that is:

$$U(x) = U(a) + (x-a)U'(a) + \frac{(x-a)^2}{2!}U''(a) + \cdots + \frac{(x-a)^{n+2}}{(n+2)!}U^{(n+2)}(a).$$

4. Numerical Examples

Example (1)

Consider the linear Fredholm integro-ordinary differential problem of order two:

$$U''(x) + U'(x) - U(x) = 2x^3 - 6x^2 - 18x - \frac{47}{30} + \int_0^1 (x-t)U(t)dt$$

With The boundary conditions: $U(0) = 5, U'(0) = 1,$

$U(1) = 4, U'(1) = -5.$

So that $k(x,t) = x - t, f(x) = 2x^3 - 6x^2 - 18x - \frac{47}{30}, P_0(x) = .$

-1 *and* $P_1(x) = 1$

Take h=0.1, $i = k = 2$, expand U(x) in Taylor series

$$U(x) = U(a) + (x-a)U'(a) + \frac{(x-a)^2}{2!}U''(a) + \frac{(x-a)^3}{3!}U'''(a) + \frac{(x-a)^4}{4!}U^{(4)}(a).$$

Table (1) shows a comparison between the numerical and exact solution $U(x) = 5 + x - 2x^3$

Table (1)

X	THE APPROXIMATE SOLUTION	EXACT
0	5	5
0.1	5.098000000	5.098000000
0.2	5.184000000	5.184000000
0.3	5.246000000	5.246000000
0.4	5.272000000	5.272000000
0.5	5.250000000	5.250000000
0.6	5.168000000	5.168000000
0.7	5.014000000	5.014000000
0.8	4.776000000	4.776000000
0.9	4.442000000	4.442000000
1	4.000000000	4.000000000
L.S.E	0.000000000	

Example (2)

Consider the linear Fredholm integro-ordinary differential equations of order one:

$$U'(x) + U(x) = x^2 + \frac{5}{4}x + 1 + \int_0^1 xtU(t)dt$$

With boundary conditions

$U(0) = 1, U(1) = 2$

$k(x,t) = xt, f(x) =$

In this case $x^2 + \frac{5}{4}x + 1$ *and* $P_0(x) = 1$.

Take h=0.1, $i = k = 2$, expand U(x) in Taylor series:

$$U(x) = U(a) + (x-a)U'(a) + \frac{(x-a)^2}{2!}U''(a) + \frac{(x-a)^3}{3!}U'''(a)$$

Comparison of the exact solution $u(x)=1+x^2$ and the Taylor approximate solution with h=0.1 is illustrated in Table (2) with the corresponding least square errors.

Table (2)

X	THE APPROXIMATE SOLUTION	EXACT
0	1	1
0.1	1.010000000	1.010000000
0.2	1.040000000	1.040000000
0.3	1.090000000	1.090000000
0.4	1.160000000	1.160000000
0.5	1.250000000	1.250000000
0.6	1.360000000	1.360000000
0.7	1.490000000	1.490000000
0.8	1.640000000	1.640000000
0.9	1.810000000	1.810000000
1	2.000000000	2.000000000
L.S.E	0.000000000	

Example (3)

Given the following linear Volterra integro-ordinary differential equation of order two:

$$U''(x)+U'(x)=$$

$$1+x-\frac{x^3}{2}-\frac{x^5}{8}+\int_0^x xtU(t)dt, \qquad 0\le x\le 1$$

With initial condition $U(0)=1, U'(0)=0$, and exact solution

$$U(x)=1+\frac{x^2}{2}$$

.In this case

$$k(x,t)=xt, f(x)=$$

$$1+x-\frac{x^3}{2}-\frac{x^5}{8}, P_0(x)=0 \; and \; P_1(x)=1$$

Table (3) shows a comparison between the true and the approximate solution with h=0.1.

Table (3)

X	THE APPROXIMATE SOLUTION	EXACT
0	1	1
0.1	1.005000000	1.005000000
0.2	1.019999999	1.020000000
0.3	1.044999999	1.045000000
0.4	1.079999999	1.080000000
0.5	1.124999999	1.125000000
0.6	1.179999999	1.180000000
0.7	1.244999999	1.245000000
0.8	1.319999999	1.320000000
0.9	1.404999999	1.405000000
1	1.499999999	1.500000000
L.S.E	1.192×10^{-11}	

Example (4):

Consider the following problem:

$$U'(x)+U(x)=x^3+3x^2-5x-3-$$

$$(2x-\frac{5}{2}x^2+\frac{1}{4}x^4)\sin x+\int_0^x \sin xU(t)dt, \qquad 0\le x\le 1$$

With initial condition $U(0)=2$, and exact solution $U(x)=2-5x+x^3$. Take h=0.1, expand U(x) in Taylor series the results are listed in Table (4) with the exact solution.

Table (4)

X	THE APPROXIMATE SOLUTION	EXACT
0	2	2
0.1	1.501000000	1.501000000
0.2	1.008000000	1.008000000
0.3	0.527000001	0.527000000
0.4	0.064000002	0.064000000
0.5	-0.374999999	-0.375000000
0.6	-0.783999999	-0.784000000
0.7	-1.156999999	-1.157000000
0.8	-1.487999986	-1.488000000
0.9	-1.770999980	-1.771000000
1	-1.999999972	-2.000000000
L.S.E	3.880×10^{-8}	

Conclusions

Taylor expansion approach can be applied to solve linear Volterra and Fredholm integro-ordinary differential equations of order n with satisfactory results. The main problem in this method is undoubtedly the derivative, so this problem can be circumvented by using central differences operator with a suitable value increment h . Therefore the proposed method is convenient for computer programming.

References

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الحل العددي لمعادلات تفاضلية-تكاملية خطية بأستخدام متسلسلة تايلر

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المستخلص

متسلسلة تيلر طبقت لحل نوعين مهمين من المعادلات التكاملية-التفاضلية الخطية ذات رتب عليا : فريدهولم وفولتيرا . في هذه الطريقة تم تعويض متسلسلة تيلر مكان الدالة المجهولة بعد أن تم اشتقاق طرفي المعادلة التكاملية-التفاضلية بالنسبة إلى x كما واستخدمت الفروقات المركزية والقواعد التربيعية لإيجاد القيم التقريبية للمشتقات والتكاملات الناتجة على التوالي . تم استخدام طريقة الحذف لكاوس لحل المعادلات الجبرية الناتجة . كتبت البرامج الخاصة بهذه الطريقة باستخدام لغةMATLAB. وقدمت أمثلة لتوضيح نتائج هذه الطريقة .