Semigroup ideal in Prime Near-Rings with Derivations

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Abstract:

In this paper we generalize some of the results due to Bell and Mason on a near-ring N admitting a derivation D , and we will show that the body of evidence on prime near-rings with derivations have the behavior of the ring. Our purpose in this work is to explore further this ring like behavior. Also, we show that under appropriate additional hypothesis a near-ring must be a commutative ring.

Key words and phrases: Prime Near-ring , Semiprime Near–ring, Semigroup ideal, derivation.
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Introduction:

Throughout this paper N will denote a zero – symmetric left near-ring with multiplicative center $Z(N)$. An additive mapping $D:N\rightarrow N$ is called derivation if $D(xy)=xD(y)+D(x)y$, for all $x,y\in N$. A near–ring N is called a zero symmetric if $0x=0$, for all $x\in N$. Further an element $x\in N$ for which $D(x)=0$ is called a constant. For $x,y\in N$, the symbol $[x,y]$ will denote the commutator $xy-yx$, while the symbol (x,y) will denote the additive –group commutator $x+y-x-y$.The derivation D will be called commuting if $[x,D(x)]=0$, for all $x\in N$.According to Bell and Mason[1], and Bell and Kappe [2], a near-ring N is said to be prime if $xNy=0$, for $x,y\in N$ implies $x=0$ or $y=0$. A non empty subset I of N will be called a semigroup ideal if $IN\subseteq I$ and $NI \subseteq I$ .indeed , there are several results(see for example [1,2,3,4,5,6]) asserting that the existence of a suitably-constrained derivation on a prime near-ring forces the near–ring to be a ring as for terminologies used here without mention ,we refer to G.Pliz[7].

Results:

We need the following lemmas.

Lemma 1([8]). Let N be a prime near-ring and I be a nonzero semigroup ideal of N .If $(I,+)$ is abelian , then $(N,+)$ is abelian.

Proof. Since$(I,+)$ is abelian , we have $u+v=v+u$, for all $u,v\in I$.Taking xu instead of u and yu instead of v , where $x,y \in N$,we obtain $xu+yu=yu+xu$.Then , we get $(x+y-x-y)u=0$, for all $u\in I$, and $x,y\in N$. It means that $(x,y)I=0$.Since I is a semigroup ideal of N , we get $(x,y)NI=0$.Since N is prime near-ring and I is a nonzero, we get $(x,y)=0$, for all $x,y\in N$.Thus, $(N,+)$ is abelain.

Lemma 2. Let D be an arbitrary derivation on the near-ring N and I be a semigroup of N, then $(aD(b)+D(a)b)c=aD(b)c+D(a)bc$, for all $a,b\in I$ and $c\in N$.

Proof. For all $a,b\in I$ and $c\in N$,we get
$D((ab)c)=abD(c)+D(ab)c$

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$= abD(c)+(aD(b)+D(a)b)c$

On the other hand,

$D(a(bc))=aD(bc)+D(a)bc$

$=abD(c)+aD(b)c+D(a)bc$

For these two expressions of $D(abc)$, for all $a,b\in I$ we obtain that $(aD(b)+D(a)b)c=aD(b)c+D(a)bc$.

Lemma 3. Let D be a derivation on near-ring N and I be a semigroup ideal of N, suppose $u\in I$ is not a left zero divisor .If $[u,D(u)]=0$, then $(x.u)$ is a constant for every $x\in I$.

Proof. From $u(u+x)=u^2+ux$, apply D for both sided we have $D(u(u+x))=D(u^2+ux)$.Expanding this equation , we have $uD(u+x)+D(u)(u+x)=uD(u)+D(u)u+uD(x)+D(u)x$. Which reduces to $uD(x)+D(u)u=D(u)u+uD(x)$, for $u,x\in I$. By using the hypothesis $[u,D(u)]=0$, this equation is expressible as $u(D(x)+D(u)-D(x)-D(u))=0=u(D(x,u))$.Since u is not a left zero divisor , we get $D((x,u))=0$.Thus ,(x,u) is a constant for every $x\in I$.

Theorem 1. Let N be a near-ring and I be a semigroup ideal of N have no a nonzero divesors of zero.If N admits a nonzero derivation D which is commuting on I, then $(N,+)$ is abelain.

Proof. Let c be any additive commutator in I .Then , by application Lemma(3) yields that c is a constant .For any $x\in I$, xc is also an additive commutator in I. Hence, also a constant .Thus,$0=D(xc)=xD(c)+D(x)c$. First term in this equation equal zero , we get $D(x)c=0$, for all $x\in I$ an additive commutator c in I .Since $D(x)\neq 0$, for some $x\in I$ and I have no nonzero divisors of zero , we get $c=0$, for all additive commutator c in I .Hence, $(I,+)$ is abelian .By Lemma(1), we get $(N,+)$ is abelian.

Lemma4. Let N be a prime near-ring and I be a semigroup ideal of N.

(i) If z is a nonzero element in $Z(N)$, then z is not a zero divisor.

(ii)If there exists a nonzero element z of $Z(N)$ such that $z+z\in Z(N)$, then $(I,+)$ is abelain.

Proof.(i) If $z\in Z(N)\backslash\{0\}$, and $zx=0$ for some $x\in I$.Left multiplicative this equation by b,where $b\in N$, we get $bzx=0$.Since N is multiplicative with center $Z(N)$, we get $zbx=0$, for all $b\in N,x\in I$.Hence, $zNx=0$.Since N is a prime near-ring and z is a nonzero element , we get $x=0$.

(ii) Let $z\in Z(N)\backslash\{0\}$ be an element , such that $z+z\in Z(N)$, and let $x,y\in I$, such that $(x+y)(z+z)=(z+z)(x+y)$.Hence , $xz+xz+yz+yz=zx+zy+zx+zy$. Since $z\in Z(N)$, we get $zx+zy=zy+zx$. Thus , $z(x+y-x-y)=0$.Left multiplicative this equation by b where $b\in N$, we get $bz(x,y)=0$ for all $x,y\in I$ and $b\in N$ .Since N is multiplicative with center $Z(N)$, we get $zb(x,y)=0$, for all $x,y\in I$,$b\in N$. Hence ,$zN(x,y)=0$.Since N is a prime near-ring and z is a nonzero element, we get $(x,y)=0$ for all $x,y\in I$. Thus $(I,+)$ is abelain.

Lemma 5. Let D be a nonzero derivation on a prime near-ring N and I be a nonzero semigroup ideal of N .Then $xD(I)=0$ implies $x=0$ and $D(I)x=0$ implies $x=0$, where $x\in N$.

Proof. Let $xD(I)=0$.For any $r\in N,s\in I$ .Then $xD(sr)=0$ for $x,r\in N$ and $s\in I$. Thus ,$xsD(r)+xD(s)r=0$, the second term in this equation equal zero by the hypothesis , we get $xsD(r)=0$ for $x,r\in N$ and $s\in I$ .Hence $xID(r)=0$.Since I is a Semigroup ideal of N, we get $xIND(r)=0$.Since N is a prime near –ring and I is a nonzero Semigroup ideal , D is a nonzero derivation of N , we get $x=0$.By similar way, we can show that if $D(I)x=0$, for all $x\in N$ implies that $x=0$.

Lemma 6. Let N be a prime near – ring and I be a nonzero semigroup ideal of N. If N is a 2-torsion free and D is a derivation on N such that $D^2(I)=0$, then $D(I)=0$.

Proof. For arbitrary $x,y\in I$ .Suppose D is a nonzero derivation , we have $0=D^2(xy)=D(D(xy))=D(xD(y)+D(x)y)= xD^2(y)+D(x)D(y)+ D(x)D(y)+D^2(x)y$.By the hypothesis , we get $2D(x)D(y)=0$ for all $x,y \in I$. Since N is a 2-torsion free , we get $D(x)D(y)=0$.thus, $D(x)D(I)=0$, for all $x\in I$.By lemma (5), we get $D=0$.

Lemma 7. Let N be a prime near-ring and I be a nonzero semigroup ideal of N and D be a nonzero derivation on N.If $D((x,y))=0$, for all $x,y\in I$, then $(I,+)$ is abelain.

Proof. Suppose that $D((x,y))=0$, for all $x,y\in I$ .Taking ux instead of x and uy instead of y ,where $u\in I$, we get $0=D((ux,uy))=D(u(x,y))=uD((x,y))+D(u)(x,y)$, for all $x,y,u\in I$.By the hypothesis we have $D(u)(x,y)=0$, for all $x,y,u\in I$.Hence, $D(I)(x,y)=0$.By lemma (5) , we get $(x,y)=0$, for all $x,y\in I$. Thus ,$(I,+)$ is abelian.

Lemma 8. Let N be a prime near-ring and I be a nonzero semigroup ideal of N. If I is a commutative then N is a commutative ring .

Proof. For all $a,b\in I$.$[a,b]=0$. Taking ax instead of a and by instead of b, where $x,y\in N$, we get $[ax,by]=0$, since I is a commutative and Semigroup ideal of N, we have $0=axby-byax=baxy-byax=abxy-abyx=ab[x,y]$, for all $a,b\in I$, $x,y\in N$.Thus $ab[x,y]=0=I^2[x,y]$.Since I is a Semigroup ideal of N , we get $I^2N[x,y]=0$, for all $x,y\in N$.Since N is a prime near-ring and I is a nonzero, we get $[x,y]=0$, for all $x,y\in N$.Hence , N is a commutative ring.

Lemma 9. Let N be a prime near – ring admits a nonzero derivation D and I be a semigroup ideal of N such that $D(I)\subseteq Z(N)$, then $(I,+)$ is abelian .If N is a 2-torsion free and $D(I)\subseteq I$, then I is a central ideal .

Proof.Since $D(I)\subseteq Z(N)$ and D is a nonzero derivation .There exists a nonzero element x in I , such that $z=D(X) \in Z(N)\backslash\{0\}$. And , $z+z=D(x)+D(x)=D(x+x) \in Z(N)$. Hence $(I,+)$ is abelian by lemma (4). (ii). Using hypothesis , for any $a,b\in I$ and $c\in N$. $cD(ab)=D(ab)c$.

By using Lemma (2), we have $caD(b)+cD(a)b=aD(b)c+D(a)bc$.

Using $D(I)\subseteq Z(N)$ and $(I,+)$ is abelian, we get $caD(b)+D(a)cb=acD(b)+D(a)bc$. So, we have $[c,a]D(b)=D(a)[b,c]$, for all $a,b\in I$, $c\in N$ .Suppose that I is not a central ideal .Choosing $b \in I$ and $c \in N$ such that $[b,c]\neq 0$ .And since $D(I)\subseteq I$, let $a=D(x) \in Z(N)$, where $x\in I$, we get $[c,D(x)]D(b)=D^2(x)[b,c]$, for all $x,b\in I$, $c\in N$.Then $D^2(x)[b,c]=0$, for all $x\in I$ .By Lemma (4) (i) the central element $D^2(x)$ can not be a nonzero divisor of zero , then we conclude that $D^2(x)=0$, for all $x\in I$.By lemma(6) , we get $D(x)=0$, this contradiction with D is a nonzero derivation on N,we get $[b,c]=0$, this contradiction with assumption .Hence , I is a central ideal.

Theorem 2. Let N be a prime near-ring admits a nonzero derivation D and I be a semigroup ideal of N such that $D(I)\subseteq Z(N)$, then $(N,+)$ is abelian .If N is a 2-torsion free and $D(I)\subseteq I$, then N is a commutative ring.

Proof. By Lemma (9) , we have $(I,+)$ is abelain .By Lemma(1) , we get $(N,+)$ is abelain .Now, assume N is a 2-

torssion free. By application Lemma(9), we get I is a central ideal .Thus I is a commutative. By lemma(8), we get N is a commutative ring.

Theorem 3. Let N be a prime near-ring admitting a nonzero derivation D and I be a nonzero semigroup ideal of N such that $[D(I),D(I)]=0$, then $(N,+)$ is abelian .If N is a 2-torsion free and $D(I)\subseteq I$, then N is commutative ring.

Proof. By the hypothesis , for all $x,y,t\in I$, we have $D(t+t)D(x+y)=D(x+y)D(t+t)$. Hence, $D(t)D(x)+D(t)D(y)+D(t)D(x)+D(t)D(y) =D(x)D(t)+ D(x)D(t)+ D(y)D(t)+ D(y)D(t)$.By application the hypothesis in this equation , we get $D(t)D((x,y))=0$, for all $x,y,t\in I$, Thus , $D(I)D((x,y))=0$.By using Lemma(5), we get $D((x,y))=0$, for all $x,y\in I$ .By Lemma(7), we get (I,+) is abelian .By Lemma(1) , we obtain $(N,+)$ is abelian .Assume that N is a 2-torsion free , by the assumption $[D(I),D(I)]=0$, we have $D(D(x)y)D(z)=D(z)D(D(x)y)$, for all $x,y,z\in I$. Hence , by Lemma (2), we get

$D(x)D(y)D(z)+D^2(x)yD(z)=D(z)D(x)D(y)+D(z)D^2(x)y$

Since $(N,+)$ is abelian and by the assumption, we conclude that

$D^2(x)yD(z)=D^2(x)D(z)y$, for all $x,y,z\in I$ …..(*)

Replacing y by yt, where $t\in N$, in equation (*) , and using equation(*), we get

$D^2(x)ytD(z)=D^2(x)D(z)yt=D^2(x)yD(z)t$, for all $x,y,z\in I$, $t\in N$.

Then , we get $D^2(x)y[t,D(z)]=0$, for all $x,y,z\in I$,$t\in N$.Hence , $D^2(x)I[t,D(z)]=0$, for all $x,z\in I$, $t\in N$. Since I is a semigroup ideal of N , we have $D^2(x)IN[t,D(z)]=0$ .Since N is a prime near-ring , we get $D^2(x)I=0$ or $[t,D(z)]=0$ for all $x,z\in I$, $t\in N$. If $D^2(x)I=0$, then since I is a nonzero semigroup ideal and N is a prime near-ring , we get $D^2(x)=0$, by Lemma(6), we get $D(x)=0$, this a contradiction with D is a nonzero on N. So, $[t,D(z)]=0$ for all $z\in I$, $t\in N$, we get $D(I)\subseteq Z(N)$.Thus , N is a commutative ring .By application Theorem(2).

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المثاليات شبه الاولية على الحلقات المقتربة الاولية مع الاشتقاق

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الخلاصة:

في هذا البحث سنعمم بعض النتائج التي ظهرت عند الباحثين بل وماسون على الحلقة المقتربة N بوجود الاشتقاق D وكذلك سوف نرى حجم الادلة على الحلقة المقتربة الاولية مع الاشتقاق تملك سلوك الحلقة. الغرض من هذا العمل هو اكتشاف تلك الحلقة التي لها نفس السلوك وكذلك سوف نرى تحت اضافة مناسبة من الفرضيات ان الحلقة المقتربة يجب ان تكون حلقة ابدالية.