Product of Conjugacy Classes of the Alternating Group A_n

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Abstract:

For a nonempty subset X of a group G and a positive integer m , the product of X , denoted by X^m ,is the set

$$X^m = \left\{ \prod_{i=1}^{m} x_i \quad : x_i \in X \right\}$$

That is , X^m is the subset of G formed by considering all possible ordered products of m elements form X. In the symmetric group S_n, the class C_n (n odd positive integer) split into two conjugacy classes in A_n denoted C_n^+ and C_n^- . C^+ and C^- were used for these two parts of C_n. This work we prove that for some odd n ,the class C of 5- cycle in S_n has the property that $C^{\frac{n-3}{2}} = A_n$ $n \geq 7$ and C^+ has the property that each element of C^+ is conjugate to its inverse, the square of each element of it is the element of C^-, these results were used to prove that C^+ C^- = A_n exceptional of I (I the identity conjugacy class), when n=5+4k , k>=0.

Key words: conjugacy classes ,split, Alternating Group, Product

Introduction:

The product of conjugacy classes of the symmetric group S_n is found to be a linear combination of conjugacy classes of the symmetric group with integer coefficients,[1]. Dvir in [2] developed a theory of the product of the conjugacy classes in A_n and S_n, $n \geq 5$ which satisfy $C^3 = A_n$. Brenner proved that for the conjugacy class C of n- cycle of the Alternating group A_n, CCC=A_n, n=4k-1 and CC=A_n, n=4k+1>5.[3] Lamia H.R. in [3] proved that for n=5+8k, $C^+ C^+ C^+ = A_n$,. In the Symmetric group S_n, the class C $\in S_n$ splits into two conjugacy classes of the same order C^+ and C^-, these splitting happens with respect to the conjugator if it is even or odd. For each x,y $\in$ C, $\delta \notin A_n$ we have $\delta^{-1} x \delta = y \in C^-$ when $x \in C^+$. After splitting we can see if $x^{-1} \in C^+$ or not by using the formula [(n-l)/2] which gives the number of transposition in the standard conjugator, if it is even then $x^{-1} \in C^+$ and otherwise $x^{-1} \notin C^+$,[4].

The Conjugacy Classes of the Alternating Group A_n

In this section, some basic definitions and fundamental results which are necessary for the main results are given.

Proposition (1), [4]:
Let C_α be the conjugacy class of S_n. Then C_α splits into two A_n-classes of equal order if and only if n>1 and the non-zero parts of α are pair wise different and odd. In all other cases C_α does not split. We denote these two split classes C^+ and C^-.

Proposition (2), [2]:
For n odd, the cycle (12…n) is conjugate to its inverse in A_n if and only if $n \equiv 1 \pmod 4$.

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Lemma (3), [5]:

If a finite subgroup H of a group G is the union of conjugacy classes in G, then H is a normal subgroup of G.

Results:

This section depends on two type of conjugacy classes of the symmetric group S_n

1- The conjugacy class C of type 51^{n-5}

For S_7 the class C of type 51^{n-5} has the property that $CC=A_7$. The same for S_9 the class $CCC=A_9$ and so on.

2- The conjugacy class which split into two classes C^+ and C^-.

By using proposition (2.1), the class C of length n (n odd positive integer) split into two classes, this split happened with respect to n (proposition 2.2), for example in A_5

C^+=
[(12345),(15432),(13254),(14523),(13425),(15243), 12453), (12534) , (14352), (13542), (15324), (14235)]

C^-= [(13524), (14253), (12435), (15342), (14532), (12354), (14325) , (15423), (13245), (15234), 13452), (12543)] .

these two type of classes were used to prove some results of this paper as in the following:

Lemma (1)

Let C be the class of the 5- cycle in S_7, then $C^2=A_7$.

Proof

Since each element of the class C of S_5 contains in the class C of S_7

Then CC contains some elements of A_7

For the other elements of A_7 we have:

(12345)(12346)=(136)(245) $\in A_7$

(12345)(12463)=(1452)(36) $\in A_7$

(12345)(12467)=(1452367) $\in A_7$

(12345)(14267)=(167)(23)(45) $\in A_7$

So CC contains all elements of A_7

Then $C^2=A_7$.

Lemma (2)

Let $n \geq 7$ be odd ,let C be the class of 5- cycle in S_n then $C^{\frac{n-3}{2}} = A_n$.

Proof

The proof is by induction on n:

For n=7 we have from lemma (1), $C^2 = A_7$,which implies that CC is the normal subgroup of S_7.Since n is odd , assume its true for n-2 which is odd, $C^{\frac{n-5}{2}} =A_{n-2}$, $C^{\frac{n-5}{2}}$ is the normal subgroup of S_{n-2}.Now we prove it true for n, since $C^{\frac{n-5}{2}}$ is the normal subgroup of S_{n-2} which contains all even conjugacy classes by product $C^{\frac{n-5}{2}}$ C we get the conjugacy classes of A_{n-2} as well as anther even conjugacy classes, since $C^{\frac{n-5}{2}}$ is normal subgroup so the product $C^{\frac{n-5}{2}}$ C which equal two $C^{\frac{n-3}{2}}$ is also normal subgroup , then $C^{\frac{n-3}{2}} =A_n$.

Lemma (3)

Let n=5+4k, (k=0,1,2,…), let C^+ be the class of (12…n), C^- be the class of (21…n) in A_n then $C^+ C^-$ contains the conjugacy classes of period 2.

Proof

For n=5, we have for any two elements of C^+ & C^- of A_5
(12345)(12354)=(13)(25)

So the class of type 2^2 contains in C^+C^-

The same for n=9, we have
(123456789) C^+ ,(123985674) C^- (12345 $\in$89)(123985674) $\in$ (13)(29)(46)(57)

For any two elements of the conjugacy classes C^+,C^- we have

(12…n)(n…54123)= (13)(24)

(12…n) (n…85674123) = (13)(2n)(46)(57)

So C^+C^- contains the conjugacy classes of period two.

Lemma (4)

Let n=5+4k . Let C^+ be the class of (12…n), C^- be the class of (21…n) in A_n then C^+ C^- contains the conjugacy class of a 3- cycle.

Proof

By the same way of lemma (3)

We have ,for any two elements of C^+, C^- we have

(12…n)(n…54312)=(1n2)

Then C^+ C^- contains the conjugacy class of 3- cycle

Theorem (5)

In the Alternating group A_n ,if n=5+4k ≥5 ,(k=0,1,…), then C^+ C^- = A_n exceptional of I.

Proof

Let C be the conjugacy class of A_n of length n which split into C^+ and C^- .Since by lemma [2.2]

Then the inverse of each element of C^+ is belonging to C^+.So C^+ C^- doesn't contain the identity.

Now we prove that C^+ C^- contain all other conjugacy classes of A_n.

The prove is by induction on n , for n=5, let a= (12345) $\in C^+$ for each element of C^- we have :

(13452) $\in C^-$ $\longrightarrow$ (12345) (13452) = (24) (35) $\in A_5$

(15342) $\in C^-$ $\longrightarrow$ (12345) (15342) = (243) $\in A_5$

(14253) $\in C^-$ $\longrightarrow$ (12345) (14253) = (15432) $\in A_5$

(13524) $\in C^-$ $\longrightarrow$ (12345) (13524) = (14253) $\in A_5$

Since all conjugacy classes contain in A_5 ,this implies that C^+ C^-= A_5 except for I. The theorem is valid for n=5.

Now for n-2, since n is odd so n-2 is odd , assume its true for n-2 which implies that $C^+ C^- = A_{n-2}$.

We want to prove it for n

Let S $\neq$ I be a permutation of A_n. If the largest cycle in S is a 2- cycle, then by using [lemma 3] , $C^+ C^- =A_n$.

If S is a single 3-cycle, then by using [lemma 4] $C^+C^- =A_n$.

Next we must show that S $=k_1k_2$ and k_1, k_2 are n-cycles belonging to different classes in A_n. To see this write d_1= (123…n-2) , d_2 = (135…n-2 24…n-3) in different classes in A_{n-2} . By using t= (n n-1 n-3) such that

$d_1 t^{-1}$=(123…n-2)(n-3 n-1 n)=(123…n-1 n n-3) $\in A_n$.

d_2 (n n-2 n-1)=(135…n-2 24…n-3)=(135…n-1 n n-2) $\in A_n$.

The two cycles displayed on the right – hand sides are in different classes in A_n, since d_1,d_2 are in different classes in A_{n-2} so the theorem follows.

Conclusions :

From this work some conclusions are drown ; they listed below:

1- The product of conjugacy classes C of type 51^{n-5} ,$C^{\frac{n-3}{2}}$ = A_n, for n n≥ 7

2- The product of split conjugacy classes C^+C^- contains the conjugacy class of period 2 when n=5+4k.

3- The product of split conjugacy classes C^+C^- contains the conjugacy class of a 3- cycle when n=5+4k.

4- The product of split conjugacy classes C^+C^- = A_n , n=5+4k ≥5, exceptional of I.

5- In future we can study the split conjugacy class $C_\alpha^{\pm}$ with the property $\alpha_i \equiv 3$ (mod 4) is even.

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حول ضرب صفوف التكافؤ بالزمرة المتناوبة A_n

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الخلاصة:

يعرف حاصل ضرب عناصر المجموعة الجزئية X في الزمرة G على أنه

$$X^m = \left\{ \prod_{i=1}^{m} x_i \quad : x_i \in X \right\}$$

صف التكافؤ C_n في الزمرة المتناظرة S_n عندما تكون n عدد فردي يتجزأ الى جزئين في الزمرة المتناوبة A_n يرمز لهذين الجزئين C_n^-, C_n^+ استخدمنا C^-, C^+ لهذين النوعين من الصفوف في هذا البحث تم التوصل الى ان :-

$= A_n$ $n \geq 7$ C is the class of 5- cycle of $S_n C^{\frac{n-3}{2}}$

A_n exceptional of I(I identity conjugacy class) when n=5+4k k>=0=C^+ C^-