A Solution of Second Kind Volterra Integral Equations Using Third Order Non-Polynomial Spline Function

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Abstract:

In this paper, third order non-polynomial spline function is used to solve 2nd kind Volterra integral equations. Numerical examples are presented to illustrate the applications of this method, and to compare the computed results with other known methods.

Keywords: Volterra integral equation, non-polynomial spline function, cubic spline function.

Introduction:

Many problems of mathematical physics can be started in the form of integral equations. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. Numerical simulation in engineering science and in applied mathematics has become a powerful tool to model the physical phenomena, particularly when analytical solutions are not available then very difficult to obtain. Therefore, the study of integral equations and methods for solving them are very useful in application. In recent years, there has been a growing interest in the Volterra integral equations arising in various fields of physics and engineering [1], Lima,P. and Diogo, T. in(1997) [2] presented an extrapolation method to find numerical solution of VIE's with weakly singular kernel .Rashidinia, J. and Zarebnia, M. in(2008) [3] used sinc function method to find the numerical solution of linear VIE's of the second kind**.** Bizar ,J. and Eslami , M. in(2011) [4] presented Homotopy Perturbation and Taylar series method for solving VIE's of second kind . Maleknejad, K. Hashmizadeh, E.andEzzati, R. in(2012)[5] studied a new approach to find the numerical solution of VIE's by using Bernsteins Approximation.

Many researchers have used non-polynomial spline functions approach to find the solution of differential equations. Ramadan , M.A. El-Danaf , T. and Abdaal F. E.I. in(2007) [6] Presented an application of the non-polynomial spline function to find the solution of the burgers equation. Zarebnia M. Hoshyar , M. and Sedahti, M. in(2011)[7] Presented a numerical solution based on non-polynomial cubic spline function is used for finding the solution of boundary value problem.

Recently they use of non-polynomial spline method to solve some kinds of integral equation like: Hossinpour , A. in (2012)[8] presented the solution of integral differential equation by non-polynomial spline functions and quaderature formula. AL-Khalidi S.H.H. in (2013)[9] presented algorithms for solving Volterra integral equations using non-polynomial spline functions**.** Rahman M. M., Hakim, M. A., Kamrul Hasan, M., Alam M. K. and Nowsher, L. in (2012)[10] use the numerical method

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Chebyshev polynomial for solving Volterra second kind .

Third order Non-polynomial Spline Function:

Let the linear Volterra integral equation (VIE) of the second kind be in the form

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt\ , \quad a \le x \le b \quad \dots (1)$$

Consider the partition $\Delta = \{t_0, t_1, t_2, \dots, t_n\}$ of $[a,b] \subset \mathbb{R}$. Let $S(\Delta)$ denote the set of piecewise continuous polynomials on subinterval $I_i = [t_i, t_{i+1}]$ of partition Δ. In this work, third order non-polynomial spline function will be used for finding approximate solution of VIE's of the second kind. Consider the grid point t_i on the interval $[a,b]$ as follows:

$$a=t_0< t_1< t_2< \dots <t_n=b \quad \dots (2)$$

$$t_i=t_0+ih\ , i=0,1,\dots,n \quad \dots (3)$$

$$h = \frac{b-a}{n} \quad \dots (4)$$

where n is a positive integer. The suggested third order non-polynomial spline function is:

$$P_i(t) = a_i cosk(t-t_i) + b_i sink(t-t_i) + c_i(t-t_i) + d_i(t-t_i)^2 + e_i(t-t_i)^3 + m_i \quad \dots (5)$$

where a_i, b_i, c_i, d_i, e_i and m_i are constants to be determined, and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method.

Let $u(t)$ be the exact solution of equation (1) and $S_i(t)$ be an approximate to $u_i=u(t_i)$ obtained by the segment $P_i(t)$.

The following relations must be satisfied:

$$P_i(t_i) = a_i + m_i = u(t_i) \approx S_i(t_i)$$

$$P'_i(t_i) = kb_i + c_i = u'(t_i) \approx S'_i(t_i)$$

$$P''_i(t_i) = -k^2 a_i + 2d_i = u''(t_i) \approx S''_i(t_i)$$

$$p'''_i(t_i) = -k^3 b_i + 6e_i = u'''(t_i) \approx S'''_i(t_i)$$

$$P''''_i(t_i) = k^4 a_i = u''''(t_i) \approx S''''_i(t_i)$$

$$p'''''_i(t_i) = k^5 b_i = u'''''(t_i) \approx S'''''_i(t_i)$$

then we can obtain the values of a_i, b_i, c_i, d_i, e_i and m_i as follows:

$$a_i = \frac{1}{k^4} u''''(t_i) \approx S''''_i(t_i) \quad \dots (6)$$

$$b_i = \frac{1}{k^5} u'''''(t_i) \approx S'''_i(t_i) \quad \dots (7)$$

$$c_i = u'(t_i) - kb_i \approx S'_i(t_i) - kbi \quad \dots (8)$$

$$d_i = 1/2[u''(t_i) + k^2 a_i] \approx 1/2[S''_i(t_i) + k^2 a_i] \quad \dots (9)$$

$$e_i = d_i = 1/6[u'''(t_i) + k^3 b_i] \approx 1/6[S''_i(t_i) + k^3 b_i] \quad \dots (10)$$

$$m_i = u(t_i) - a_i \approx S_i(t_i) - a_i \quad \dots (11)$$

for $i=0,1,\dots,n$.

Method of Solution:

To solve the linear VIE's of the second kind eq.(1), we differentiate it five times with respect to x and evaluate them at $x=a$:

$$u_0 = u(a) = f(a) \quad \dots (12)$$

$$u'_0 = u'(a) = f'(a) + k(a,a)u(a) \quad \dots (13)$$

$$u''_0 = u''(a) = f''(a) + \left(\frac{\partial k(x,t)}{\partial x}|_{t=x}\right)_{x=a} u(a) + \frac{dk(x,x)}{dx}|_{x=a} u(a) + k(a,a)u'(a) \quad \dots (14)$$

$$u'''_0 = u'''(a) = f'''(a) + \left(\frac{\partial^2 k(x,t)}{\partial x^2}|_{t=x}\right)_{x=a} u(a) + \left(\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}|_{t=x}\right)\right)_{x=a} u(a) + \left(\frac{\partial k(x,t)}{\partial x}|_{t=x}\right)_{x=a} u'(a) + \frac{d^2 k(x,x)}{dx^2}|_{x=a} u(a) + 2\frac{dk(x,x)}{dx}|_{x=a} u'(a) + k(a,a)u''(a) \quad \dots (15)$$

$$u_0^{(4)} = u^{(4)}(a)$$
$$= f^4(a) + \left[[\frac{\partial^3 k(x,t)}{\partial x^3}]_{t=x}\right]_{x=a} u(a) + \left[\frac{d}{dx}\left(\frac{\partial^2 k(x,t)}{\partial x^2}\right)_{t=x}\right]_{x=a} u(a)$$
$$+ \left(\frac{\partial^2 k(x,t)}{\partial x^2}\right)_{t=x} u'(a) + \left[\frac{d^2}{dx^2}\left[\frac{\partial k(x,t)}{\partial x}\right]_{t=x}\right]_{x=a} u(a)$$
$$+ 2\left[\frac{d}{dx}\left[\frac{\partial k(x,t)}{\partial x}\right]_{t=x}\right]_{x=a} u'(a) + \left(\left[\frac{\partial k(x,t)}{\partial x}\right]_{t=x}\right)_{x=a} u''(a)$$
$$+ \left(\frac{d^3 k(x,x)}{dx^3}\right)_{x=a} u(a) + 3\left(\frac{d^2 k(x,x)}{dx^2}\right)_{x=a} u'(a) + 3\left(\frac{dk(x,x)}{dx}\right)_{x=a} u''(a)$$
$$+ k(a,a)u'''(a) \qquad \ldots(16)$$

$$u_0^{(5)} = u^{(5)}(a) = f^5(a) + \left[[\frac{\partial^4 k(x,t)}{\partial x^4}]_{t=x}\right]_{x=a} u(a) + \left[\frac{d}{dx}\left(\frac{\partial^3 k(x,t)}{\partial x^3}\right)_{t=x}\right]_{x=a} u(a) +$$
$$\left(\left[\frac{\partial^3 k(x,t)}{\partial x^3}\right]_{t=x}\right)_{x=a} u'(a) + \left[\frac{d^2}{dx^2}\left[\frac{\partial^2 k(x,t)}{\partial x^2}\right]_{t=x}\right]_{x=a} u(a) +$$
$$2\left[\frac{d}{dx}\left[\frac{\partial^2 k(x,t)}{\partial x^2}\right]_{t=x}\right]_{x=a} u'(a) + \left(\left[\frac{\partial^2 k(x,t)}{\partial x^2}\right]_{t=x}\right)_{x=a} u''(a) +$$
$$\left[\frac{d^3}{dx^3}\left[\frac{\partial k(x,t)}{\partial x}\right]_{t=x}\right]_{x=a} u(a) + 3\left[\frac{d^2}{dx^2}\left[\frac{\partial k(x,t)}{\partial x}\right]_{t=x}\right]_{x=a} u'(a) +$$
$$3\left[\frac{d}{dx}\left[\frac{\partial k(x,t)}{\partial x}\right]_{t=x}\right]_{x=a} u''(a) + \left[\frac{\partial k(x,t)}{\partial x}\right]_{t=x=a} u'''(a) + \left[\frac{d^4 k(x,x)}{d^4 x}\right]_{x=a} u(a) +$$
$$4\left[\frac{d^3 k(x,x)}{dx^3}\right]_{x=a} u'(a) + 6\left[\frac{d^2 k(x,x)}{dx^2}\right]_{x=a} u''^{(a)} + 4\left[\frac{dk(x,x)}{dx}\right]_{x=a} u'''(a) +$$
$$k(a,a)u''''(a) \qquad \ldots(17)$$

Therefore, we approximate the solution of equation (1) using equation (5) in the following algorithm (VIENPS):

Algorithm (VIENPS)

To find the approximate solution of eq.(1), first we select positive integer n, and perform the following steps:

Step 1: Set h=(b-a)/n; $t_i = t_0 + ih, i = 0,1,\ldots,n$, $t_0 = a, t_n = b$ and $u_0 = f(a)$

Step 2: Evaluate a_0, b_0, c_0, d_0, e_0 and m_0 by substituting (12)-(17) in equations(6)-(11).

Step 3: Calculate $P_0(t)$ using step 2 and equation (5) for *i=0.*

Step 4: Approximate $u_1 \approx P_0(t_1)$.

Step 5: For i=1 to n-1 do the following steps:

Step 6: Evaluate a_i, b_i, c_i, d_i, e_i and m_i using equations (6)-(11)and replacing $u(t_i)$ and it's derivatives by $P_i(t_i)$ and it's derivatives **.**

Step 7: Calculate $P_i(t)$ using step 6 and equation (5).

Step 8: Approximate $u_{\mathbf{i+1}} = P_i(t_{i+1})$.

Numerical Examples:

Example (1): Consider the VIE of the second kind [10]:

$$u(x) = x3^x + \int_0^x -3^{x-t}u(t)dt \quad 0 \le x \le 1$$

with exact solution $u(x) = 3^x(1-e^{-x})$.Results have been shown in Table 1, where $P_i(x)$ denote the approximate solution by the proposed method and err = $|\boldsymbol{u}(x)$ - $P_i(x)|$.

Table 1: Computed Absolute Error of Example (1) and The Result Obtained in [10]

x	Exact Solution $u(x)$	$P_i(x)$	Error	error obtain in [10]
0.1	0.106213163030966	0.106213158495688	4.535278860795522e-09	1.1600483 e-002
0.2	0.225812709291563	0.2258124145160261	2.947755368409855e-07	2.8608994e-002
0.3	0.360363539107344	0.360360129707884	3.409399463250029e-06	2.2232608e-002
0.4	0.511612377368213	0.511592928966171	1.944840204171072e-05	1.0103823e-002
0.5	0.681508888598327	0.681433578057413	7.531054091380884e-05	1.7285379e-002
0.6	0.872229243985166	0.872001001385275	2.282425998910709e-04	6.5041788e-003
0.7	1.086202425097018	1.085618339803819	5.840852931990881e-04	8.3481474e-003
0.8	1.326139582081997	1.324818967967492	1.320614114505458e-03	5.7238171e-003
0.9	1.595806680104478	1.592350411094480	2.716389950296438e-03	1.1187893-003
1.0	1.89636167648567	1.891176122008416	5.185554477256549e-03	1.1830649e-002
$\|err\|_\infty$			**5.185554477256549e-03**	**2.8608994e-002**

Example 2: Consider the VIE of the second kind [3]:

$$u(x) = 1 - x + \frac{x^2}{2} + \int_0^x t - x\, u(t)dt$$

$$0 \le x \le 1$$

With exact solution $u(x) = 1 - \sin(x)$.. Results have been shown in Table 2, where $P_i(x)$ denote the approximate solution by the proposed method.

Table 2: Computed Absolute Error of Example (2) and The Result Obtained in [3]

x	Exact Solution $u(x)$	$P_i(x)$	Error	error obtain in [3]
0.1	9.001665833531718e-01	9.001665833531718e-01	0	-
0.2	8.013306692049388e-01	8.013306692049388e-01	0	-
0.3	7.044797933386604e-01	7.044797933386604e-01	0	-
0.4	6.105816576913494e-01	6.105816576913496e-01	2.220446049250313e-16	-
0.5	5.205744613957970e-01	5.205744613957971e-01	1.110223024625157e-16	-
0.6	4.353575266049646e-01	4.353575266049647e-01	1.110223024625157e-16	-
0.7	3.557823127623090e-01	3.557823127623091e-01	1.110223024625157e-16	-
0.8	2.826439091004772e-01	2.826439091004774e-01	2.220446049250313e-16	-
0.9	2.166730903725166e-01	2.166730903725169e-01	3.330669073875470e-16	-
1.0	1.585290151921035e-01	1.585290151921039e-01	4.440892098500626e-16	-
$\|err\|_\infty$			**4.440892098500626e-16**	**3.6208210e-04**

Conclusion:

In this paper, non-polynomial spline function method for solving Volterra integral equations of the second kind is presented successfully. This idea based on the use of the VIE's and its derivatives. So it is necessary to mention that this approach can be used when f(x) and k(x,t) are analytic. The proposed scheme is simple and computationally attractive and their accuracy are high and we can execute this method in a computer simply. The numerical examples support this claim, and fig. (1,2) are plotted to show the comparison between the exact and approximate solution of these examples.

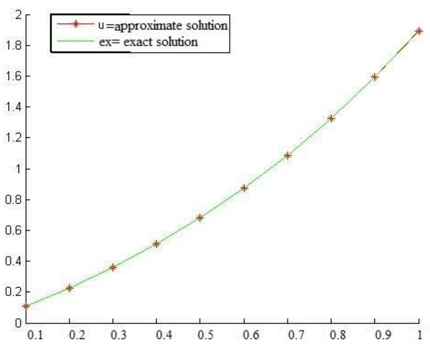

fig.1 example 1 acomparition between the exact and the approximate solution

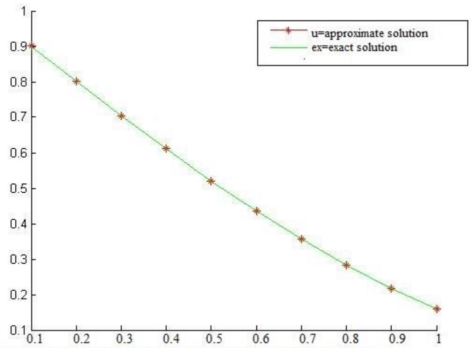

fig.2 example 2 a comparition between the exact and the approximate solution

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حل لمعادلات فولتيرا التكاملية من النوع الثاني باستخدام دالة الثلمة الغير متعددة الحدود من الدرجة الثالثة

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الخلاصة:

في هذا البحث تم استخدام دالة الثلمة الغير متعددة الحدود من الدرجة الثالثة لإيجاد حل عددي تقريبي لمعادلات فولتيرا التكاملية من النوع الثاني. تم اعطاء امثلة عددية لتوضيح تطبيق الطريقة، كما تم مقارنة النتائج مع طرق اخرى معروفة.

الكلمات المفتاحية:معادلة فولتيرا التكاملية، دالة الثلمة الغير متعددة الحدود ، دالة الثلمة من الدرجة الثالثة.