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Notes on Traces of a Symmetric Generalized (σ, τ)-Biderivations and Commutativity in Prime Rings

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Abstract:

Let R be a 2-torision free prime ring and σ, $\tau \in Aut(R)$. Furthermore, G: $R\times R\rightarrow R$ is a symmetric generalized (σ, τ)-Biderivation associated with a nonzero (σ, τ)-Biderivation D. In this paper some certain identities are presented satisfying by the traces of G and D on an ideal of R which forces R to be commutative.

Key words: Prime rings, Symmetric generalized (σ, τ)-Biderivation, Trace of biadditive mappings, Commuting mappings.

Introduction:

Throughout R will be a ring with center $Z(R)$ and σ, $\tau \in Aut(R)$. A ring R is called 2-torsion free, if $2x=0$, $x \in R$, implies $x=0$. The symbol $[x, y]$ represent the commutator $xy - yx$ and xoz to the Jordan product $xy + yx$. The following identities of commutator and Jordan product are useful to prove our theorems which are hold for any $x,y, z \in R$.

- $[xz, y] = [x, y]z + x[z, y]$
- $[x, zy] = [x, z]y + z[x, y]$
- $[x, zy]_{\sigma,\tau} = \tau(z)[x, y]_{\sigma,\tau} + [x, z]_{\sigma,\tau}\,\sigma(y)$
- $[xz, y]_{\sigma,\tau} = x[z, y]_{\sigma,\tau} + [x, \tau(y)]\,z = x[z, \sigma(y)] + [x, y]_{\sigma,\tau}z$
- $(xz) \circ y = x(z \circ y) - [x, y]z = (x \circ y)z + x[z, y]$
- $x \circ (zy) = (x \circ z)y - z[x, y] = z(x \circ y) + [x, z]y$

Recall that R is prime if for any $a,b \in R$, $aRb =\{0\}$ implies $a=0$ or $b=0$ and semiprime if for any $a \in R$, $aRa =\{0\}$ implies $a=0$. It's clear that every prime ring is a semiprime but the converse in general is not true. Let S be a sub ring of R. A mapping η: $S\longrightarrow R$ is said to be centralizing on S if $[\eta(x), x]\in Z(R)$, for all $x \in S$. Furthermore, η is called commuting whenever $[\eta(x), x] =0$, for all $x \in S$ (see [1]). It's known that the most important studies in the ring theory is the one that characterizing commutativity of prime and semiprime rings .So over the last thirty years a lot of works have been done concerning the commutativity of rings admitting suitably constrained additive or biadditive mappings such as automorphisms, derivations and Biderivations acting on appropriate subsets of the rings (see, e.g. [1, 2, 3, 4]). A mapping $\mathcal{B}$: $R\times R\rightarrow R$ is said to be symmetric if $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ holds for all pairs $x,y \in R$. A mapping f: $R \longrightarrow R$ defined by $f(x) = \mathcal{B}(x , x)$, where $\mathcal{B}$ is a symmetric mapping will be called the trace of $\mathcal{B}$. It obvious that in case $\mathcal{B}$ is a symmetric mapping which is also biadditive (i.e., additive in both arguments), the trace of $\mathcal{B}$ satisfies $f(x+y)= f(x) +2\mathcal{B}(x,y)+ f(y)$, for all $x,y \in R$ (see [5]).

The notion of symmetric Biderivation was introduced by G. Maksa in [6]. A symmetric biadditive mapping $D(. , .): R\times R\longrightarrow R$ is called symmetric Biderivation if $D(xy, z)=D(x, z) y + xD(y, z)$ holds for all $x,y,z \in R$. The notion of additive commuting mapping is closely connected with notion of Biderivations, that is every commuting additive mapping $f:S \longrightarrow R$ gives rise to Biderivation $D: S\times S\longrightarrow R$ defend by $D(x, y)= [f(x), y]$, for all $x, y \in S$ (see [7]).

In 2007 Y. Ceven, and M. A. Öztürk in [7] introduce the concept of symmetric (σ, τ)- Biderivation as follows: A symmetric biadditive mapping $F(. , .):R\times R\longrightarrow R$ is said to be a symmetric (σ, τ)-Biderivation if $F(xy, z)= F(x, z)\sigma(y)+\tau(x) F(y, z)$, for all $x,y,z \in R$. It's clear that in this case the relation $F(x,yz) = F(x, y) \sigma(z)+ \tau(y) F(x, y)$ is also satisfied for all $x,y,z \in R$. In 2010 M. Ashraf introduced in [8] the notion of symmetric generalized (σ, τ)-Biderivation as follows: A symmetric biadditive mapping $G(. , .):R\times R \longrightarrow R$ is symmetric generalized (σ, τ)-Biderivation if there exists symmetric (σ, τ)-Biderivation D such that $G(xz, y) =G(x, y) \sigma(z) + \tau(x)D(z, y)$, for all $x, y, z \in R$.

Motivated by theses works mentioned above, in this paper we continue the line of investigation regarding the relationship between commutativity of a rings and the existences of certain specific types of traces of symmetric (σ, τ)-Biderivation and a symmetric generalized (σ, τ)-Biderivation.

1. ***Some Preliminaries***

We shall list the following lemmas which will be used extensively to prove our theorems.

Lemma (1.1): [9]

Let R be a semiprime ring, $\mathcal{I}$ an ideal of R. If $\mathcal{I}$ is a commutative as a ring, then $\mathcal{I} \subset Z(R)$. In addition if R is prime then R must be commutative.

Lemma (1.2): [10]

Let R be a prime ring of characteristic different from 2 and $\mathcal{I}$ be a nonzero ideal of R. let a, b be afixed elements of R. if $axb+bxa=0$ is fulfilled for all $x \in \mathcal{I}$, then either $a=0$ or $b=0$.

2. ***The Main results***

We begin our discussion with the following theorem which extends Lemma (2.4) that obtained in [11] to a symmetric (σ, τ)-biderivation.

Theorem (2.1):

Let R be a 2-torision free prime ring and U be a nonzero left ideal of R. If R admits a symmetric (σ, τ)-Biderivation $F: R\times R\longrightarrow R$ with $f(u) =0$, for all $u \in U$, where f is the Trace of F, then either F is the zero on R or R is commutative.

Proof: By the hypothesis, we have:

$f(u) =0$, for all $u \in U$. (1)

Linearization (1), using this and the 2-torision freeness, we obtain:

$F(u, v)=0$, for all $u,v \in U$. (2)

Consequently

$F(ru, sv)=0$, for all $u,v \in U$ and $r,s \in R$.

Expanding this term in two different ways, we get:

$0= F(ru, sv)$

$= F(r, sv) \sigma(u) + \tau(r)F(u, sv)$

$= F(r, s) \sigma(v) \sigma(u)+ \tau(s)F(r, v)\sigma(u)+ \tau(r)F(u, s)\sigma(v) + \tau(r)\tau(s)F(u, v)$.

On the other hand, we have:

$0= F(ru, sv)$

$= F(ru, s) \sigma(v) + \tau(s)F(ru, v)$

$= F(r, s) \sigma(u) \sigma(v)+ \tau(r)F(u, s)\sigma(v)+ \tau(s)F(r, v)\sigma(u) + \tau(s)\tau(r)F(u, v)$.

Comparing the two expressions of $F(ru, sv)$, we arrive because of (2) to:

$F(r, s) \sigma(v) \sigma(u) = F(r, s) \sigma(u) \sigma(v)$, for all $u,v \in U$ and $r,s \in R$.

That is

$F(r, s) [\sigma(v), \sigma(u)]= 0$, for all $u,v \in U$ and $r,s \in R$. (3)

The substitution ts for s in (3) and using (3) yields that:

$F(r, s)\ \sigma(t)\ [\sigma(v), \sigma(u)] = 0 = 0$, for all $u, v \in U$ and $r, s, t \in R$.

Equivalently

$\sigma^{-1}(F(r, s))\ R\ [v, u] = 0$, for all $u, v \in U$ and $r, s \in R$.

Using the primeness of R gather together with the automorphismity of σ, we conclude that either F is the zero on R or:

$[v, u] = 0$, for all $u, v \in U$.

This forces U to be a commutative ideal of R, hence R is commutative by Lemma(1.1).

Corollary (2.2): [11, Lemma (2.4)]

Let R be a 2-torision free prime ring and $\mathcal{J}$ be a nonzero ideal of R. If D is a symmetric Biderivation such that $D(x, x) = 0$, all $x \in \mathcal{J}$.then either $D = 0$ or R is commutative.

Theorem (2.3):

Let R be a 2-torision free prime ring and U a nonzero ideal of R. Suppose $G: R\times R\to R$ is a symmetric generalized Biderivation associated with a nonzero Biderivation D such that the Traces d and g of D and G respectively satisfies one of the following:

i - $[d(u), g(v)] = [u, v]$, for all $u, v \in U$.

ii - $[d(u), g(v)] = u o v$, for all $u, v \in U$.

then either R is commutative or $G(U,U) \subseteq Z(R)$.

Proof:

(i) Suppose that for d and g, we have:

$[d(u), g(v)] = [u, v]$, for all $u, v \in U$. (1)

Linearization of the relation (1) with respect to u gives:

$[d(u), g(v)] + [d(\omega), g(v)] + 2[D(u, \omega), g(v)] = [u, v] + [\omega, v]$, for all $u, v, \omega \in U$.

According to (1), the above relation reduces because of the 2-torisionity free of R to:

$[D(u, \omega), g(v)] = 0$, for all $u, v, \omega \in U$. (2)

Putting ωu for ω in (2) leads to:

$[D(u, \omega), g(v)]\ u + D(u, \omega)[u, g(v)] + \omega[d(u), g(v)] + [\omega, g(v)]\ d(u) = 0$, for all $u, v, \omega \in U$.

The above relation becomes in view of relation (2):

$D(u, \omega)[u, g(v)] + \omega[d(u), g(v)] + [\omega, g(v)]\ d(u) = 0$, for all $u, v, \omega \in U$. (3)

The substitution $u\omega$ for ω in the relation (3) and using (3), we see:

$d(u)\ \omega\ [u, g(v)] + [u, g(v)]\ \omega\ d(u) = 0$, for all $u, v, \omega \in U$.

An application of Lemma (1.2) implies that either $d(u) = 0$, for all $u \in U$ or $[u, g(v)] = 0$, for all $u, v \in U$. So U is the union of the following two sub groups of U:

$\mathcal{M} = \{ u \in U: d(u) = 0\}$

$\mathcal{N} = \{ u \in U: [u, g(v)] = 0\}$

Since a group cannot be the set theoretic union of two it's proper subgroups, hence either $U = \mathcal{M}$ or $U = \mathcal{N}$. If $U = \mathcal{M}$, that $d(u) = 0$, for all $u \in U$, then an application of corollary (3.2) implies that R is commutative. Otherwise, $U = \mathcal{N}$, that is::

$[u, g(v)] = 0$, for all $u, v \in U$. (5)

The linearization of (5) with respect to v gives because of (5) and the 2-torision freeness:

$[u, G(v, \omega)] = 0$, for all $u, v, \omega \in U$. (6)

Replacing u by ur in (6) and using (6), we find:

$u\ [r, G(v, \omega)] = 0$, for all $u, v, \omega \in U$ and $r \in R$.

Equivalently

$U\ R\ [r, G(v, \omega)] = 0$, for all $u, v, \omega \in U$ and $r \in R$.

Since U is a nonzero ideal of R, we conclude that $G(U,U) \subseteq Z(R)$.

(ii) Suppose that for any $u, v \in U$, we have:

$[d(u), g(v)] = u o v$. (7)

The Linearization of the above relation with respect to u we arrive because of (7) and 2-torisionity free of R to:

$[D(u, \omega), g(v)] = 0$, for all $u, v, \omega \in U$.

This relation is similar to relation (2) in part *(i)*, hence using the same technique as used in the proof of the part *(i)*, we obtain the required result.

Theorem (2.4):

Let R be a 2-torision free prime ring and U a nonzero ideal of R. Suppose $G: R\times R\to R$ is a symmetric generalized Biderivation associated with a nonzero Biderivation D satisfies that $d(u) \circ g(v)= [u, v]$, for all $u, v \in U$, where d and g are the Traces of D and G respectively, then either R is commutative or d is commuting on U.

Proof: Suppose that the Traces d and g satisfy:

$d(u) \circ g(v)= [u, v]$, for all $u, v \in U$. (1)

Linearization of the relation (1) with respect to v leads to:

$d(u) \circ g(v) + d(u) \circ g(\omega) + 2\, d(u) \circ G(v, \omega)= [u, v] + [\omega, v]$, for all $u, v \in U$.

The above relation reduces because of (1) and 2-torisionity free of R to:

$d(u) \circ G(v, \omega)= 0$, for all $u, v, \omega \in U$. (2)

Substituting ωv for ω in (2) gives:

$d(u) \circ G(v, \omega)v + d(u) \circ \omega d(v) = 0$, for all $u, v, \omega \in U$.

According to (2), the last relation becomes:

$d(u) \circ \omega d(v) = 0$, for all $u, v, \omega \in U$.

That is

$d(u)\, \omega\, d(v) + \omega d(v)d(u) = 0$, for all $u, v, \omega \in U$. (3)

Left multiplication of (3) by u, we get:

$ud(u)\, \omega\, d(v) + u\omega d(v)d(u)= 0$, for all $u, v, \omega \in U$. (4)

Putting $u\omega$ instead of ω in (3) yields that:

$d(u)\, u\omega\, d(v) + u\omega d(v)d(u)= 0$, for all $u, v, \omega \in U$. (5)

By subtracting the relation (4) from (5), we find:

$[d(u), u]\, \omega\, d(v)= 0$, for all $u, v, \omega \in U$.

But U is an ideal of R, we have:

$[d(u), u]\omega R\, d(v)= 0$, for all $u, v, \omega \in U$.

By the primeness property of R we conclude that either $d(v)= 0$ or $[d(u), u]\omega = 0$, for all $u, v, \omega \in U$.

If $d(v)= 0$, for all $v \in U$, then an application of theorem (2.1) implies that R is commutative.

On another hand, if $[d(u), u]\omega = 0$, for all $u, \omega \in U$, then:

$[d(u), u]\, R\, \omega = 0$, for all $u, \omega \in U$.

Since U is a nonzero ideal of R, we get:

$[d(u), u] = 0$, for all $u \in U$.

This means that d is commuting on U.

Theorem (2.5):

Let R be a 2-torision free prime ring and U a nonzero ideal of R. Suppose $F: R\times R\to R$ is a symmetric generalized (σ, τ)-Biderivation associated with a nonzero (σ, τ)-Biderivation D such that $[f(u), u]_{\sigma,\tau}=0$, for all $u \in U$, where f is the Trace F, then R is commutative.

Proof: Suppose that for any $u \in U$, we have:

$[f(u), u]_{\sigma,\tau}=0$. (1)

Taking $u+v$ for u in (1), using (1) leads to:

$[f(u), v]_{\sigma,\tau} + 2[F(u, v), u]_{\sigma,\tau} + 2[F(u, v), v]_{\sigma,\tau} + [f(v), u]_{\sigma,\tau} =0$, for all $u, v \in U$.

Substituting $2u$ instead of u, and then comparing the relation so obtained with the above one, we get:

$[f(u), v]_{\sigma,\tau} + 2[F(u, v), u]_{\sigma,\tau} =0$, for all $u, v \in U$. (2)

Putting vu for v in (2), we find:

$\tau(v)[f(u), u]_{\sigma,\tau} +[f(u), v]_{\sigma,\tau}\, \sigma(u) + 2F(u, v)[\sigma(u), \sigma(u)]+ 2[F(u, v), u]_{\sigma,\tau}\, \sigma(u) + 2\tau(v)[d(u), u]_{\sigma,\tau}+ 2[\tau(v), \tau(u)]d(u)= 0$, for all $u, v \in U$.

Where d is the trace of D, according to the relations (1), (2) and the 2-torisionity free of R, we get:

$\tau(v)[d(u), u]_{\sigma,\tau}+ [\tau(v), \tau(u)]\, d(u)= 0$, for all $u, v \in U$. (3)

Replacing v by $v\omega$ in (3), we see:

$\tau(v)\tau(\omega)[d(u), u]_{\sigma,\tau}+[\tau(v), \tau(u)]\tau(\omega)\, d(u) + \tau(v)[\tau(\omega), \tau(u)]\, d(u)= 0$, for all $u, v, \omega \in U$.

The above relation reduces because of (3), to:

$[\tau(v), \tau(u)]\tau(\omega)\, d(u) = 0$, for all $u, v, \omega \in U$.

That is

$[v, u]\ \omega\ \tau^{-1}(d(u)) = 0$, for all $u,v,\omega \in U$. (4)

Consequently

$[v, u]\ R\,\omega\tau^{-1}(d(u)) = 0$, for all $u,v,\omega \in U$.

Since a group cannot be the set theoretic union of two it's proper subgroups, either $[v, u] =0$, for all $u,v \in U$ or $U\,\tau^{-1}(d(u))=0$, for all $u \in U$.

If $[v, u] =0$, for all $u,v \in U$ yields that U is commutative ideal, consequently R is commutative by Lemma (1.1). Otherwise

$U\,\tau^{-1}(d(u))=0$, for all $u \in U$.

Equivalently

$U\,R\,\tau^{-1}(d(u))=0$, for all $u \in U$.

Since U is a nonzero ideal of R and τ is an automorphisms of R, we conclude that $d(u)=0$, for all $u \in U$.

Hence R is commutative by theorem (2.1).

Theorem (2.6):

Let R be a 2-torision free prime ring, U a nonzero ideal of R. Suppose $F: R\times R\rightarrow R$ is a symmetric generalized (σ, τ)-Biderivation associated with a nonzero (σ, τ)-Biderivation D such that $f(u)\sigma(u)= \tau(u)\, d(u)$, for all $u \in U$, where f and d are the Traces of F and D respectively, then either R is commutative or d is commuting on U.

Proof: Suppose that the Traces f and d satisfy:

$f(u)\sigma(u)= \tau(u)d(u)$, for all $u \in U$. (1)

Taking $u+v$ instead of u in (1) and using (1), we get:

$f(u)\sigma(v) + 2F(u, v)\sigma(v) + 2F(u, v)\sigma(u) + f(v)\sigma(u) = -\tau(v)d(u) -2\tau(u)D(u, v) - 2\tau(v)D(u, v) -\tau(u)d(v)$

Writing $-u$ for u, then combining the above relation with the relation so obtained, we find:

$f(u)\sigma(v) + 2F(u, v)\sigma(u) = -\tau(v)d(u) - 2\tau(u)D(u, v)$, for all $u,v \in U$. (2)

Replacing v by vu in (2) leads to:

$f(u)\sigma(v)\sigma(u)+ 2F(u, v)\,\sigma^2(u) + 2\tau(v)d(u)\sigma(u)= -\tau(v)\tau(u)d(u) -2\tau(u)D(u, v)\sigma(u)- 2\tau(u)\tau(v)d(u)$

In view of (2), the last relation reduces to:

$-\tau(v)d(u)\sigma(u)- 2\tau(u)D(u, v)\sigma(u) + 2\tau(v)d(u)\sigma(u) = -\tau(v)\tau(u)d(u) -2\tau(u)D(u, v)\sigma(u)- 2\tau(u)\tau(v)d(u)$

That is

$\tau(v)\ \mathbf{(}d(u)o\sigma(u)\mathbf{)} = - 2\tau(u)\tau(v)d(u)$, for all $u, v \in U$. (3)

The substitution ωv for v in (3) gives:

$\tau(\omega)\tau(v)\mathbf{(}d(u)o\sigma(u)\mathbf{)} = -2\tau(u)\tau(\omega)\tau(v)d(u)$, for all $u, v, \omega \in U$.

According to (3) and the 2-torisionity free of R, the above relation becomes:

$\tau(\omega)\tau(u)\tau(v)d(u)= \tau(u)\tau(\omega)\tau(v)d(u)$, for all $u, v, \omega \in U$.

That is

$[\tau(\omega), \tau(u)]\ \tau(v)\ d(u)=0$, for all $u, v, \omega \in U$.

Equivalently

$[\omega, u]\ U\,\tau^{-1}(d(u))=0$, for all $u, \omega \in U$.

This relation is similar to relation (4) in theorem (2.6), hence moving in the same manner as in the proof of the theorem (2.6) our result gets completed.

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إشارات حول دوال الأثر لثنائية المشتقات-(σ, τ) المُعَمَّمة المتناظرة والتبادلية في الحلقات الأولية

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الخلاصة:

لتكن R حلقة أولية إلتوائها 2 وكل من σ, τ تشاكلات تقابلية ذاتية. فضلا عن ذلك $G: R\times R \rightarrow R$ هي دالة ثنائية المشتقات-(σ, τ) المُعَمَّمة المتناظرة مترافقة مع دالة ثنائية المشتقات-(σ, τ) المتناظرة D. في هذا البحث قدمنا بعض المتطابقات التي تحققها دوال الأثر لكل من G و D على مثالي ما في R التي تفرض على الحلقة R الخاصية الإبدالية.

الكلمات المفتاحية: الحلقات الأولية، ثنائية المشتقات-(σ, τ) المُعَمَّمة المتناظرة، دوال الأثر للدوال ثنائية الخطية، الدوال التبادلية.