An Approximate Solution of some Variational Problems Using Boubaker Polynomials

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Abstract:
In this paper, an approximate solution of nonlinear two points boundary variational problem is presented. Boubaker polynomials have been utilized to reduce these problems into quadratic programming problem. The convergence of this polynomial has been verified; also different numerical examples were given to show the applicability and validity of this method.

Keywords: Boubaker polynomials, variational problems, nonlinear programming.

Introduction:

Many problems in mathematical physics and engineering are connected with the calculus of variations which is concerned with finding extrema for functional opposed to function. Usually functionals are defined by definite integrals, with boundary conditions and smoothness requirements, which appear in the problem formulation (1).

Many researchers have worked in this field, M. Razzaghia, S. Yousefi used in 2000 Legendre wavelets to solve variational problems(2). Abdulaziz O., and others in 2008 applied Homotopy-perturbation method to obtain approximate analytical solutions to variational problems(3). Dixit S. and Singh K. and others in 2010 used Bernstein orthonormal polynomials of degree six (4). Ordokhani Y. in 2011 applied Walsh-hybrid functions for solving the problems of variational, using a combination of block-pulse functions and Walsh functions then with Bernoulli polynomials in another paper (5,6). Najeeb S., Abdalelah A. in 2012, utilized Chebyshev wavelets for solving some variational problems(7). Mohammadi R. and Sadat A. in 2015 developed a new method using quadratic polynomial Spline (1). Also Mohammadi R., Zahedi M. and Bayat Z. in 2015, developed an exponential Spline method for solving calculus of variations(8).

This paper can be outlined as follows: in Section 2 Boubaker polynomials are defined with its recurrence relation. In Section3, the application of Boubaker polynomials for solving variational problems, with some numerical examples have been presented.

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Section 4 the convergence test for the introduced method has been studied, then the conclusion.

2- Boubaker polynomials(9, 10):

Boubaker polynomials have been applied first by Boubaker et al. to deal with heat equation in physical model, and then several papers have been applied with different applications in physics, applied sciences...etc.

Boubaker polynomial (Bo) is presented by the following equation:

$$Bo_k(z) = \sum_{r=0}^{\xi(k)} \left[\frac{(k-4r)}{(k-r)} C_{k-r}^r \right] (-1)^r z^{k-2r} \quad ...(1)$$

With

$$C_{k-r}^r = \frac{(k-r)!}{r!(k-2r)!}, \qquad k = 0,1,2,...$$

and $\xi(k) = \left\lfloor \frac{k}{2} \right\rfloor = \frac{2k+((-1)^k-1)}{4}$

$$Bo_0(z) = 1, \quad Bo_1(z) = z,$$
$$Bo_2(z) = z^2 + 2, ...$$

and a recursive relation of this polynomial has given as follows:

$$Bo_j(z) = zBo_{j-1}(z) - Bo_{j-2}(z), \qquad \text{for } j > 2$$

Also the Boubaker's operational matrix of differentiation and integration has been deduced before (see (11)).

3-Application of Boubaker polynomials for solving variational problems:

We demonstrate the application of Boubaker polynomials to solve some variational problem.

The series approximate of f^* of f is

$$f \cong f^* = \sum_{i=0}^{n} a_{in} Bo_{in} = a^T Bo(t) \quad ... (2)$$

where $a = [a_{0n}, a_{1n}, ..., a_{nn}]^T$ and $Bo(t) = [Bo_{0n}, Bo_{1n}, ..., Bo_{nn}]^T$

Consider the case $n=5$, then equation (2) becomes

$$f = \sum_{i=0}^{5} a_{i5} Bo_{i5} = a^T Bo(t) \quad ... (3)$$

Hence $a = [a_{05}, a_{15}, ..., a_{55}]$ and $Bo(t) = [Bo_{05}, Bo_{15}, ..., Bo_{55}]^T$

Differentiating equation (3) with respect to t, to get

$$\dot{f} = a^T \dot{Bo}(t) \, ... (4)$$

where $a = [a_{05}, a_{15}, ..., a_{55}]$ and $\dot{Bo}(t) = [\dot{Bo}_{05}, \dot{Bo}_{15}, ..., \dot{Bo}_{55}]^T$

then substituting in variational problem to find f.

Example 1: Using first order functional with boundary conditions

$$J(z) = \int_0^1 (\dot{z}^2 + t\dot{z} + z^2) dt \quad ... (5)$$

$z(0) = 0$, $z(1) = \frac{1}{4}$ (Boundary conditions) … (6)

The exact solution is $z(t) = \frac{1}{2} + c_1 e^t + c_2 e^{-t}$

where $c_1 = \frac{2-e}{4(e^2-1)}$ and $c_2 = \frac{e-2e^2}{4(e^2-1)}$

We approximate the variable $z(t)$ using Boubaker polynomial

$z(t) = a_{05} Bo_{05}(t) + a_{15} Bo_{15}(t) + a_{25} Bo_{25}(t) + ...$ …(7)

$z(t) = a^T Bo(t)$ …(8)

where $a = [a_{05}, a_{15}, a_{25}, a_{35}, a_{45}, a_{55}]$ and $Bo(t) = [Bo_{05}, Bo_{15}, Bo_{25}, Bo_{35}, Bo_{45}, Bo_{55}]^T$

By differentiation equation (8), we obtain

$$\dot{z}(t) = a^T \dot{Bo}(t) \quad ... (9)$$

Substituting equations (8) and (9) into (5), we get:

$$J(z) = \int_0^1 \left[a^T \dot{Bo}(t) \dot{Bo}^T(t) a + a^T t \dot{Bo}(t) \right. $$
$$\left. + a^T Bo(t) Bo^T(t) a \right] dt$$

let $H =$
$2 \int_0^1 \left[\dot{Bo}(t) \dot{Bo}^T(t) + Bo(t) Bo^T(t) \right] dt$ ... (10)

$$q^T = \int_0^1 t \dot{Bo}^T(t) dt \quad ... (11)$$

The equations (10) and (11) can be written as

$$J(z) = \frac{1}{2} a^T H a + q^T a \quad ... (12)$$

with boundary conditions equation (6)
$z(0) = a^T Bo(0) = 0$, and $z(1) = a^T Bo(1) = 1/4$
subject to $Fa - b = 0$, where

$$F = \begin{pmatrix} Bo^T(0) \\ Bo^T(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 20 & -2 & 0 \\ 1 & 1 & 32 & -1 & -3 \end{pmatrix},$$

$$b = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}.$$

The optimal values of unknown parameters a^* can be obtained by using Lagrange multiplier equation,

$a^* = -H^{-1} a + H^{-1} F^T (F H^{-1} F^T)^{-1} (F H^{-1} a + b)$,
then
a_i^*, where $i=0,1,2,3,4,5$ are
$a^* = [0.46144119, \quad 0.37468906, \quad -0.24982508, 0.07515296, -0.01910448, 0.00202181]^T$

A comparison between the approximate $z(t)$ using Boubaker polynomial and exact solution is shown in Table (1). The solution has been illustrated in Fig.1 by using Matlab.

Table 1. comparison between estimated values and exact values of $z(t)$

t	$z_{app.}$	z_{exact}	$\lvert z_{exact} - z_{app.} \rvert$
0	0	0	0
0.1	0.04195064	0.04195072	$8*10^{-8}$
0.2	0.07931744	0.07931714	$3*10^{-7}$
0.3	0.11247342	0.11247322	$2*10^{-7}$
0.4	0.14175064	0.14175081	$1.7*10^{-7}$
0.5	0.16744257	0.16744291	$3.4*10^{-7}$
0.6	0.18980653	0.18980668	$1.5*10^{-7}$
0.7	0.20906613	0.20906593	$2*10^{-7}$
0.8	0.22541368	0.22541340	$2.5*10^{-7}$
0.9	0.23901264	0.23901272	$8*10^{-8}$
1	0.24999999	0.25	$1*10^{-8}$
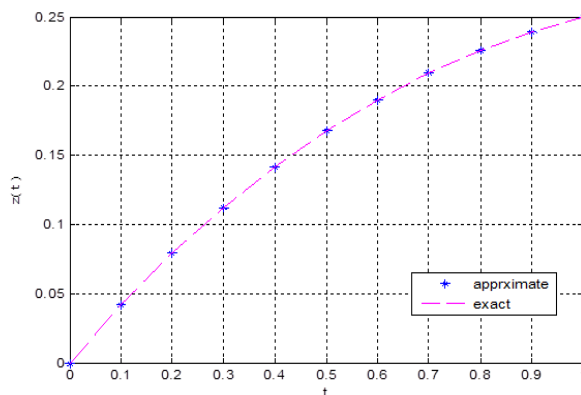
Figure 1.Shows approximate and exact solutions for example1

Example2: Using first order quadratic problem with boundary conditions

$$J(z) = \int_0^1 (\dot{z}^2 + z^2) dt \quad ... (13)$$

$z(0) = 0$, $z(1) = 1$ (boundary conditions) … (14)

The exact solution is $z(t) = \frac{e^t - e^{-t}}{e - e^{-1}}$.

As in example 1, we can compute the optimal solution of variational problem using Boubaker polynomial. The results are shown in Table (2), and illustrated in Fig.2.

Table 2.comparison between estimated values and exact values of $z(t)$

t	$z_{app.}$	z_{exact}	$|z_{exact}\text{-}z_{app.}|$
0	0	0	0
0.1	0.08523382	0.08523370	$1.2*10^{-7}$
0.2	0.17132009	0.17132045	$2.6*10^{-7}$
0.3	0.25912154	0.25912183	$2.9*10^{-7}$
0.4	0.34951677	0.34951660	$1.7*10^{-7}$
0.5	0.44340990	0.44340944	$4.6*10^{-7}$
0.6	0.54174031	0.54174007	$2.4*10^{-7}$
0.7	0.64549236	0.64549262	$2.6*10^{-7}$
0.8	0.75570506	0.75570548	$4.2*10^{-7}$
0.9	0.87348179	0.87348169	$1*10^{-7}$
1	0.99999999	1	$1*10^{-8}$
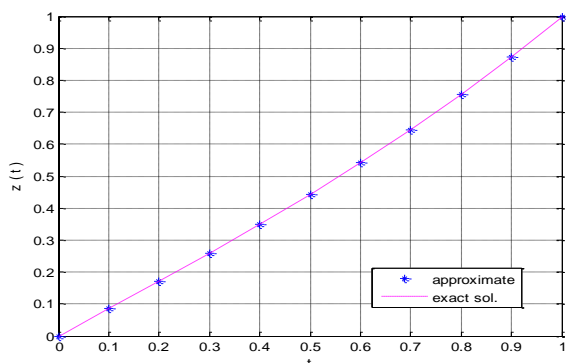
Figure2. Shows approximate and exact solutions for example 2

From the above, the results represent a good approximation to the exact solution, which indicates that Boubaker polynomials are very efficient in solving such types of variational problems.

4-The convergence test of Boubaker polynomials

Using Boubaker polynomial, the state are expanded as follows

$$z_j(t) = \sum_{i=0}^{\infty} a_{ij} Bo_i(t) \quad , j = 1,2,\dots,n \quad \dots(15)$$

The series must be truncated such that the norm in eq.(15) of the reminder $\|e(t)\|$ is less than some convergence criterion t, where $e(t) = \max\{e_1(t), e_2(t), e_3(t),\dots,e_n(t)\}$, the most useful test of convergence in terms of N is proposed with L^2 norm of z_i, i=1,2,…,n .

$$\left[\int_0^{\infty} (z_i(t) - z_{iN}(t))^2 \, dt\right]^{\frac{1}{2}} < \epsilon_i \quad i = 1,2,\dots,n$$

By taking $\epsilon = \text{Max}\{\epsilon_1(t), \epsilon_2(t),\dots,\epsilon_n(t)\}$; therefore,

$\left[\int_0^{\infty}(z(t) - z_N(t))^2 \, dt\right]^{\frac{1}{2}} < \epsilon$ for all $N > N_0$ (for some value of N_0)

Since $z(t)$ is unknown, we use the better approximation $z_{N+M}(t)$, where $M \geq 1$

$$\left[\int_0^{\infty} (z_{N+M}(t) - z_N(t))^2 \, dt\right]^{\frac{1}{2}} < \epsilon$$

$$\left[\int_0^1 \left(\sum_{i=0}^{N+M} a_i Bo_{i,N}(t) - \sum_{i=0}^{N} a_i Bo_{i,N}(t)\right)^2 dt\right]^{\frac{1}{2}} < \epsilon$$

$$\left[\int_0^1 \left(\sum_{i=N+1}^{N+M} a_i Bo_{i,N}(t)\right)^2\right]^{\frac{1}{2}} < \epsilon$$

$$\left[\int_0^1 \left(\sum_{i=N+1}^{N+M} a_i Bo_{i,N}(t)\right)\left(\sum_{i=N+1}^{N+M} a_i Bo_{i,N}(t)\right) dt\right]^{\frac{1}{2}}$$

$$< \epsilon \sum_{i=N+1}^{N+M} \sum_{j=N+1}^{N+M} a_i a_j \int_0^1 Bo_i(t)Bo_j(t)dt < \epsilon$$

So the polynomial Boubaker can be reduced to the simple form $\sum_{i=N+1}^{N+M} a_i^2 < \epsilon$

Conclusion:

In this paper, Boubaker polynomials have been utilized as an evaluation solution for reducing the variational problems into quadratic programming problems. These polynomials were proved to be an efficient and powerful tool for solving variational problems. Some examples were presented to demonstrate the applicability and validity of this method; also Matlab plotting was used to illustrate the results.

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الحل التقريبي لبعض مسائل التغايرباستخدام متعددة حدود بوبكر

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الخلاصة:

في هذا البحث تم ايجاد الحل التقريبي لمسألة التغاير بواسطة استخدام متعددة حدود بوبكر، حيث تم اختزال المسألة غير الخطية الى مسألة برمجة تربيعية، و نوقش الأقتراب المنتظم لمتعددة حدود بوبكر، وأعطيت بعض الأمثلة التوضيحية لبيان كفاءة ودقة هذه الطريقة.

الكلمات المفتاحية: متعددات حدود بوبكر، مسائل التغاير، البرمجة غير الخطية.