

Solving Optimal Control Linear Systems by Using New Third kind Chebyshev Wavelets Operational Matrix of Derivative

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Received 20, December, 2012

Accepted 26, September, 2013

Abstract:

In this paper, a new third kind Chebyshev wavelets operational matrix of derivative is presented, then the operational matrix of derivative is applied for solving optimal control problems using, third kind Chebyshev wavelets expansions. The proposed method consists of reducing the linear system of optimal control problem into a system of algebraic equations, by expanding the state variables, as a series in terms of third kind Chebyshev wavelets with unknown coefficients. Example to illustrate the effectiveness of the method has been presented.

Key words: third kind Chebyshev wavelets, operational matrix of derivative, optimal control problem .

Introduction:

Wavelets are localized function which are a useful tool in many different applications: signal analysis data compression, operator analysis, PDE solving vibration analysis and solid mechanics [1-4].

One of the popular families of wavelets is Haar wavelets [5], harmonic wavelets [6], Shannon wavelets [7], Legendre wavelets [8], and Chebyshev wavelets of the first and second kinds [9,10].

The aim of the present paper is to present wavelets named third kind Chebyshev wavelets on the interval [0,1]. The related operational matrix of derivative is derived which is suitable for approximate solution of optimal control problems.

Review of Third kind Chebyshev Wavelets:

It is well known that Chebyshev polynomials of the third kind $V_n(x)$ are orthogonal with respect to the weight

function $w(x) = \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$ on the interval

[-1,1] and satisfy the following formulae [13]

$$V_0(x) = 1$$

$$V_1(x) = 2x - 1$$

$$V_2(x) = 4x^2 - 2x - 1$$

$$V_3(x) = 8x^3 - 4x^2 - 4x + 1$$

⋮

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x)$$

To obtain the recurrence relation, we have $V_n(\cos 2\theta) = \frac{\cos(2n+1)\theta}{\cos \theta}$

and

$$V_{n+1}(\cos 2\theta) = \frac{\cos(2n+1)\theta \cos 2\theta + \sin(2n+1)\theta \sin 2\theta}{\cos \theta}$$

$$V_{n-1}(\cos 2\theta) = \frac{\cos(2n+1)\theta \cos 2\theta - \sin(2n+1)\theta \sin 2\theta}{\cos \theta}$$

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When the above relations are added together, yields

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x); \quad n = 1, 2, \dots$$

$$(1) \quad V_0(x) = 1 \quad V_1(x) = 2x - 1$$

Third kind Chebyshev wavelets are defined on the interval [0,1) by [14]

$$\Psi_{n,m}^3(t) = \begin{cases} 2^{\frac{k}{2}} \check{V}_m(2^{k+1}t - 2n + 1) \frac{n-1}{2^k} \leq t < \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

(2)

where $\check{V}_m = \frac{1}{\sqrt{\pi}} V_m$

Here $\Psi_{n,m}^3 = \Psi_{n,m}^3(k, m, n, t)$ have four arguments; $k=1, 2, 3, \dots; n=1, 2, \dots, 2^k; m$ is the order for third kind Chebyshev polynomials and t is the normalized time.

The eight basis functions when $M=3, k=1$ are given by:

$$\left. \begin{aligned} \Psi_{1,0}^3 &= \sqrt{\frac{2}{\pi}} \\ \Psi_{1,1}^3 &= \sqrt{\frac{2}{\pi}} (8t-3) \\ \Psi_{1,2}^3 &= \sqrt{\frac{2}{\pi}} ((8t-3)^2 + (8t-3) - 1) \\ \Psi_{1,3}^3 &= \sqrt{\frac{2}{\pi}} ((8t-3)^3 + 2(8t-3)^2 - (8t-3) - 1) \end{aligned} \right\} 0 \leq t < 1/2$$

(3)

$$F = [f_{1,0}, f_{1,1}, \dots, f_{1,M-1}, f_{2,0}, \dots, f_{2,M-1}, \dots, f_{2^k,0}, \dots, f_{2^k,M-1}]$$

$$\Psi^3(t) =$$

$$[\Psi_{1,0}^3(t) \Psi_{1,1}^3(t), \dots, \Psi_{1,M-1}^3(t), \Psi_{2,0}^3(t), \dots, \Psi_{2,M-1}^3(t), \dots, \Psi_{2^k,0}^3(t), \dots, \Psi_{2^k,M-1}^3(t)]^T$$

(7)

Constructing Third Kind Chebyshev Wavelets

Operational Matrix of Derivative

In order to use Chebyshev wavelets of the third kind for solving optimal

$$\left. \begin{aligned} \Psi_{2,0}^3 &= \sqrt{\frac{2}{\pi}} \\ \Psi_{2,1}^3 &= \sqrt{\frac{2}{\pi}} (8t-7) \\ \Psi_{2,2}^3 &= \sqrt{\frac{2}{\pi}} ((8t-7)^2 + (8t-7) - 1) \\ \Psi_{2,3}^3 &= \sqrt{\frac{2}{\pi}} ((8t-7)^3 + 2(8t-7)^2 - (8t-7) - 1) \end{aligned} \right\} 1/2 \leq t < 1$$

(4)

A function $f(t)$ defined over [0,1) may be expanded as:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{n,m} \Psi_{n,m}^3(t)$$

(5)

where $f_{n,m} = (f(t), \Psi_{n,m}^3(t))$

If the infinite series in equation(5) is truncated, then it can be written as:

$$f(t) \cong f_{2^k, M-1} = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{n,m} \Psi_{n,m}^3(t)$$

$$= F^T \Psi^3(t)$$

(6)

where F and $\Psi^3(t)$ are $2^k M \times 1$ matrices given by:

control problem, we need to map the differential operators to the wavelets basis .

First, consider the first derivative of third kind Chebyshev polynomials

$$D_V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 & \dots & 0 \\ 4 & 2 & 6 & 0 & 0 & 0 & \dots & 0 \\ 4 & 6 & 2 & 8 & 0 & 0 & \dots & 0 \\ 6 & 4 & 8 & 2 & 10 & 0 & \dots & 0 \\ 6 & 8 & 4 & 10 & 2 & 12 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(m-2) & \dots & 2(m-4) & 2(m-1) & 2 & 2m & \dots & 0 \\ 2m & 2(m-2) & \dots & \dots & 2(m-1) & 2 & 2m & 0 \end{pmatrix}$$

m, is the order of third Chebyshev polynomials

if m odd D_V becomes

$$\frac{dv_m}{dx} = \left[\sum_{s \text{ even}}^{m-1} (m+s+1)V_s + \sum_{s \text{ odd}}^{m-2} (m-s)V_s \right] \tag{8a}$$

if m even D_V becomes

$$\frac{dv_m}{dx} = \sum_{s \text{ even}}^{m-2} (m-s)V_s + \sum_{s \text{ odd}}^{m-1} (m+s+1)V_s \tag{8b}$$

In matrix form, the relations (8) are

$$\dot{V}_m = D_V V(t) \tag{9}$$

where D_V is an $(m+1) \times (m+1)$ matrix and $\dot{V}(t)$ is the vector of Chebyshev polynomial of the third kind given by

$$V(t) = [V_0 \ V_1 \ \dots \ V_n]^T \tag{10}$$

The third kind Chebyshev wavelets operational matrix of derivative will be derived now and introduced throughout the following theorem.

Theorem (1): The first derivative of third kind Chebyshev wavelets, is given by the following relation

$$\dot{\Psi}^3(t) = D_{\Psi^3} \Psi^3(t) \tag{11}$$

where Ψ^3 is the Chebyshev wavelets vector defined in (7) and D_{Ψ^3} is the $2^{k-1}(M+1) \times 2^{k-1}(M+1)$ operational matrix of third kind Chebyshev wavelets and is defined as follows

$D_{\Psi^3} = \text{diag}(R \ R \ \dots \ R)$ in which R is $(M+1) \times (M+1)$ matrix and its (r, s) element is defined as:

if r odd R_{rs} becomes

$$R_{rs} = 2^{k+1} \left[\sum_{s \text{ even}}^{r-1} (r+s+1) \Psi_s^\beta + \sum_{s \text{ odd}}^{r-2} (r-s) \Psi_s^\beta \right]$$

if r even R_{rs} becomes

$$R_{rs} = 2^{k+1} \left[\sum_{s \text{ even}}^{r-2} (r-s) \Psi_s^\beta + \sum_{s \text{ odd}}^{r-1} (r+s+1) \Psi_s^\beta \right] \tag{12}$$

Proof:

From eq.(2), the vector $\Psi^3(t)$ can be written as

$$\Psi_r^3(t) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} V_m^*(2^{k+1}t - n) \\ 0 \end{cases} \tag{13}$$

$t \in \left[\frac{n}{2^k}, \frac{n-1}{2^k} \right]$
O.W

where $r=0,1,\dots,(2^k M) + 1$
 $m=0,1,2,\dots,M$
 $n=0,1,2,\dots,2^k - 1$

Differentiate eq. (12) with respect to t, yields.

$$\frac{d\Psi_r^3}{dt} = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} 2^k V_m^*(2^{k+1}t - n) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Therefore; from equation (13) we can conclude that:

$$\frac{d\Psi_r^3}{dt} = \sum_{i=n(M)+1}^{(n+1)(M)} a_i \Psi_i^3; \ r = 1,2, \dots, (M) \tag{15}$$

where the coefficients a_i 's, will be obtained

$\frac{d\psi_r^3}{dt} = 0$
 $r = 0, (M + 1), \dots, 2^{k-1}(M + 1)$.
 Equation (14) is concluded because $\frac{dV_0^*(t)}{dt} = 0$, consequently the first row of matrix R defined in (12) is zero. Using relation (8) in (14) leads to equation (12).

Which is the required result .

Application to Test Problem

The objective of this section is to illustrate, through of example, the wide range of applicability and the effectiveness of the proposed method.

Example:

Consider the finite time quadratic problem Minimize

$$J = \int_0^1 u^2 dt$$

The exact solution to this problem is given by
 $x_1(t) = t^3 - 3t^2 + t + 1$ and
 $u = 6t - 6$.

$$d = [-5.67907968 \quad 0.29374550 \quad 0.058749100 \quad -3.32911568 \quad 0.76373830 \quad 0.058749100 \quad]^T$$

$$4C_8 = 0.058749100 .$$

with the aid of Spectral method to equation (17), the following system of equations are obtained,

$$8C_2 + 136C_3 + 272C_4 = -5.67907968$$

$$16C_3 + 392C_4 = 0.29374550 ,$$

$$24C_4 = 0.058749100 ,$$

$$8C_6 + 136C_7 + 272C_8 = -3.32911568$$

$$16C_7 + 392C_8 = 0.76373830 ,$$

$$\psi^3(0) = [0.79788456 \quad -2.39365365 \quad 3.9894228 \quad -5.58519192 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

$$\psi^3(1) = [0 \quad 0 \quad 0 \quad 0 \quad 0.79788456 \quad 0.79788456 \quad 0.79788456 \quad 0.79788456]^T$$

In order to apply the proposed method , one first finds

The Hamiltonian equation [15]

$$H = u^2 + \lambda_1 x_2 + \lambda_2 u$$

and the adjoint equations

$$-\frac{\partial H}{\partial x_1} = \dot{\lambda}_1 \quad , \quad -\frac{\partial H}{\partial x_2} = \dot{\lambda}_2, \quad \frac{\partial H}{\partial u} = 0$$

From the above equations , we obtain the following

$$\dot{x} + \ddot{x} = 3t^2 - 5$$

(16)

Now $x(t)$ is approximated by using third kind Chebyshev wavelets with $M = 4$ and $k=1$

That is $x = C^T \Psi^3$
 then equation (16) becomes

$$C^T (D_{\Psi^3}) \Psi^3(t) = d^T \Psi^3(t) \quad (17)$$

where $d^T \Psi^3(t) = 3t^2 - 5$

here $D_{\Psi^3 \times 8} = \begin{bmatrix} R & \vdots & O \\ \dots & \vdots & \dots \\ O & \vdots & R \end{bmatrix}$

where $R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ 8 & 16 & 0 & 0 \\ 16 & 8 & 24 & 0 \end{bmatrix}$

and

The additional two equations are given by:

$$y(0) = C^T \Psi^3(0) = 1$$

$$y(1) = C^T \Psi^3(1) = 0$$

where

Solving the above system to get the values of C^T

$$C = [1.22149171 - 0.08567577 - 0.0416139458 \quad 0.00244788 \quad 0.30108914 - 0.2912 \quad 0.01223939 \quad 0.00244788]^T$$

Table (1) numerical results of the test problem

t	<i>exact solution</i> X(t)	<i>approximat solution</i> M=4	<i>absolute Error</i>
0	1	1	0.0000000
0.1	1.07100000	1.07100000	0.0000000
0.2	1.08800000	1.08800000	0.0000000
0.3	1.05700000	1.05700000	0.0000000
0.4	0.98400000	0.98400000	0.0000000
0.5	0.87500000	0.87500000	0.0000000
0.6	0.73600000	0.73600000	0.0000000
0.7	0.57300000	0.57300000	0.0000000
0.8	0.39200000	0.39200000	0.0000000
0.9	0.19900000	0.19900000	0.0000000
1	0.00000000	0.00000000	0.0000000

Conclusion:

In this paper a general formulation for the third kind Chebyshev wavelets operational matrix of derivative has been derived. Then an approximated method based on third kind Chebyshev wavelets expansions together with the operational matrix of derivative was proposed to obtain an approximate solution of optimal control problems.

The numerical results show the method is very efficient for the numerical solution of optimal control problems and only few number of Ψ^3 expansion terms are needed to obtain a good approximate solution for these problems. The operational matrix of derivative can be applied for the numerical solution of other problems such as nonlinear optimal control problems and integral equations.

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حل منظومات خطية للسيطرة المثلى باستخدام مصفوفة العمليات للمشتقات الجديدة شبيشيف الموجية من النوع الثالث

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الخلاصة :

في هذا البحث تم عرض مصفوفة العمليات للمشتقات لشبيشيف الموجية من النوع الثالث وتطبيقها لحل مسائل السيطرة المثلى باستخدام توسيع شبيشيف الموجية من النوع الثالث . تتضمن الطريقة المقترحة بتحويل النظام الخطي لمسائل السيطرة المثلى الى منظومة معادلات خطية بتوسيع متغيرات الحالة كسلسلة بدلالة لشبيشيف الموجية من النوع الثالث . تم تقديم مثال لتوضيح كفاءة الطريقة.