Open Access
Baghdad Science Journal **Vol.16(1)Supplement 2019**

DOI: http://dx.doi.org/10.21123/bsj.2019.16.1(Suppl.).0242

Hn-Domination in Graphs

Ahmed A. Omran ***Haneen Hamed Oda****

Received 3/5/2018, Accepted 8/10/2018, Published 17/3/2019

Abstract:

The aim of this article is to introduce a new definition of domination number in graphs called *hn-domination number* denoted by $\gamma_{hn}(G)$. This paper presents some properties which show the concepts of connected and independent hn-domination. Furthermore, some bounds of these parameters are determined, specifically, the impact on hn-domination parameter is studied thoroughly in this paper when a graph is modified by deleting or adding a vertex or deleting an edge.

Keywords: , Hn-domination number, Hn-dominating set, Graph.
Mathematical subject classification: 05C69

Introduction:

In this work a graph $G = (V, E)$ (G for simplicity) is a simple, finite and undirected graph. Every term which is not found here can be found in (1,2,3).

Let V and E represent vertex set and edge set respectively for graph G. Consider a vertex v belongs to V, the number of edges incident on a vertex v is called the degree of it and is denoted by $deg(v)$ with *minimum* and *maximum degree* $\delta(G)$ and $\Delta(G)$, respectively. In case $\Delta(G) = \delta(G)$, G is called *regular graph*. A *subgraph* G_1 of a graph G is graph having all of its vertices and edges, *a spanning subgraph* is a subgraph has all vertices of G. A set $I \subseteq G$ is an *independent set* or *stable set* in graph G if its vertices are not adjacent (4).

Let G be a graph. The set $D \subseteq V$ is called *dominating* if each vertex belongs to $V - D$ is adjacent to a vertex in D. The minimum cardinality of all dominating sets is called *the domination number* of G and denoted by $\gamma(G)$ (4). The first time that the concept of domination number of a graph appeared was in (5). In (6), the first survey published some result about this concept. Recently many papers have been written on domination in graphs like (7, 8, 9, 10). Here, a new definition is introduced called *hn-domination*. Some fundamental results on hn-domination are presented. Further several bounds for the hn-domination number are stated. Also, the effects on hn-domination parameter are presented when a graph is modified by deleting a vertex or deleting or adding an edge.

Department of Mathematics, College of Education for Pure Science, University of Babylon, Babylon, Iraq.
*Corresponding author: h.hamid9993@gmail.com

Results :

Definition 2.1: Let G be a graph and D is a dominating set, the set D is called an hn-dominating set if for all adjacent $u, v \in V - D$ there are two adjacent $z_1, z_2 \in D$ such that u is adjacent to z_1 and v is adjacent to z_2 (may be $z_1 = z_2$).

Definition2.2*:* Let G be a graph and D is hn-dominating set (hn-DS), then D is called minimal hn-dominating set (hn-MDS) if it has no proper hn-dominating set. (see Fig.1).

Definition2.3: The minimum cardinality of a minimal hn-dominating set is called hn-domination number and denoted by $\gamma_{hn}(G)$.

(a) $hn - DS$ (b) non $hn - DS$

Figure 1. $hn - DS$ and non $hn - DS$

Definition 2.4: *A set D is called* γ_{hn} *−set if it is hn-dominating set with cardinality* $\gamma_{hn}(G)$.

Remark 2.5: $\gamma(G) \leq \gamma_{hn}(G)$

Proposition 2.6:

i) If graph G has a spanning star subgraph, then $\gamma_{hn}(G) = \gamma(G) = 1$.

ii) Let graph G be a non null graph and has m isolated vertices, then $\gamma_{hn}(G) \geq m + 1$.

Proof: *i*) From the definition of spanning star there is a vertex such that all other vertices are adjacent with this vertex. Thus, the result is obtained.

ii) By Definition 2.1, all isolated vertices must belong to any" $hn - DS$." Therefore, if all other vertices in G that are not isolated are dominated by at least one vertex in G (in other words if there is a spanning star formed by the other vertices of G), then by using the previous step, we get the result.

Proposition 2.7: *If* $G \cong P_n$, *then* $\gamma_{hn}(G) = \left\lfloor\frac{n}{2}\right\rfloor$.

Proof: Let $\{v_i, i = 1,2,..,n\}$ be the set of the vertices that are incident from left to right in P_n. Consider

$D = \{v_{2+2i}, i = 0,1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor - 1\}$. It is clear that D is hn-DS in P_n and $|D| = \left\lfloor\frac{n}{2}\right\rfloor$, then $\gamma_{hn}(G) \le \left\lfloor\frac{n}{2}\right\rfloor$. If we assume that there is a dominating set F of vertices with $|F| = \left\lfloor\frac{n}{2}\right\rfloor - 1$, then there is at least two adjacent vertices in $V - D$. These vertices are dominated by two vertices in D that are not adjacent. Therefore, F is not $hn - DS$, thus D is the MDS. So, $\gamma_{hn}(\mathrm{G}) = \left\lfloor\frac{n}{2}\right\rfloor$.

Remark 2.8: If $G \cong C_3$, *then* $\gamma_{hn}(G) = 1$.

Proposition 2.9: $\gamma_{hn}(C_n) = \left\lceil\frac{n}{2}\right\rceil, n \ge 4$.

Proof: There are two cases depending on the number of vertices as follows:

Case 1: If n is even, then consider the set $D = \{v_{2+2i}, i = 0,1,\ldots,\frac{n}{2} - 1\}$. In the same manner in Proposition 2.7, D is a minimum $hn - DS$, so $\gamma_{hn}(\mathrm{G}) = \frac{n}{2}$.

Case 2: If n is odd, then consider the set $D_1 = \{v_{2+2i}, i = 0,1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor - 1\}$, D_1 is a dominating set in the cycle of order n. At the same time, the set D_1 is not a $hn - DS$ in G since the two vertices v_n, v_1 are adjacent in $V - D_1$ and $v_{n-1}$and v_2 are not adjacent in D_1. Thus, we must add either v_n or v_1 to the set D_1 to obtain $hn - DS$, say v_n. Therefore, $D = D_1 \cup \{v_n\}$. Again, In the same manner in Proposition 2.7, D is the minimum $hn - DS$, so $\gamma_{hn}(G) = \left\lceil\frac{n}{2}\right\rceil$. Thus, by the results of above two cases, we get the required result.

Observation 2.10: The domination number *for graphs* K_n, W_n, *and* $K_{m,n}$ is

i) $\gamma_{hn}(K_n) = 1$.

ii) $\gamma_{hn}(W_n) = 1$.

iii) $\gamma_{hn}(K_{m,n}) = 2$.

Proposition 2.11: *If a graph* G *has a* γ_{hn}*-domination, then* $|V - D| \le m \le \frac{n(n-1)}{2}$.

Proof: Let D be a γ_{hn}- set of a graph G. To prove the lower bound; we take the two induced subgraphs $\langle D\rangle$ *and* $\langle V - D\rangle$ to be null. The edges which can appear in this case are only the edges that joining between the vertices of D and $V - D$. The minimum number of edges in this case can be determined when each vertex in $V - D$ is dominated by only one vertex in D. Therefore, the minimum number of edges in this case is $|V - D|$. Now, it is obvious that the upper bound occurs when a graph G is complete. Thus, the result is calculated.

Theorem 2.12: *If a graph* G *has a* γ_{hn}*-domination, then*

i) If D *is independent, then* $|V - D| \le m \le \frac{|V-D|(|V-D|+1)}{2}$

ii) If D *is connected, then* $|V - D| + |D| - 1 \le m \le \frac{n(n-1)}{2}$

Proof:

i) In this case the lower bound in Proposition 2.11 does not change since D is an independent set. Since the two induced subgraphs $\langle D\rangle$ *and* $\langle V - D\rangle$ can still be null graphs. The upper bound occurs when $\langle V - D\rangle$ is complete. In this case all vertices in $\langle V - D\rangle$ must be adjacent to only one vertex in D. Since, if two different vertices in $V - D$ are adjacent to two different vertices in D, then by the definition of $hn - DS$, the two different vertices in D must be adjacent. Therefore, we obtain a contradiction with the hypothesis. Thus, the maximum number of edges found in the complete graph that contain the vertices of the set $V - D$ with a vertex in D. So, the required result is obtained.

ii) In this case the upper bound in the Proposition 2.11 does not change, since a graph can be complete. The lower bound occurs when the induced subgraphs $\langle D\rangle$ is a path. Since path is connected graph with minimum edges and size of path of order $|V - D|$ is $|V - D| - 1$. Therefore, we get the result.

Proposition 2.13: *If* G *be a graph has hn-domination," then for every two adjacent vertices* v_1 *and* v_2 *in* $V - D$ *, there is a cycle containing* v_1*and* v_2.

Proof: Let $v_1$and v_2 are adjacent in $V - D$.Then two cases are obtained as follows:

Case 1: if there is a vertex in D say z such that v_1 and v_2 are adjacent to z, then v_1, v_2 and z makes a cycle.

Case 2: if $\exists z_1 \ne z_2 \in D$ such that v_1 and v_2 are adjacent to z_1 and z_2 respectively, then there is a cycle of order four for these vertices.

Corollary 2.14: *If* G *is a tree, then* $V - D$ *is independent.*

Proof: By proposition 2.12 for every two adjacent vertices v_1 and v_2 in $\mathrm{V} - \mathrm{D}$, there is a cycle contains $\mathrm{v}_1$and v_2, since G has no cycle, then all vertices in $V-D$ is not adjacent . So $V-D$ is independent.

Proposition 2.15: *If a graph* $\cong G_1 \cup G_2 \cup \dots \cup G_n$ *, then* $\gamma_{hn}(G) = \gamma_{hn}(G_1) + \gamma_{hn}(G_2) + \cdots + \gamma_{hn}(G_n)$.

Proof: It is clear that every components in G has distinct hn-dominating set with hn-domination number $\gamma_{hn}(G_i)$, $i = 1, \dots, n$. So $\gamma_{hn}(G) = \gamma_{hn}(G_1) + \gamma_{hn}(G_2) + \cdots + \gamma_{hn}(G_n)$.

Proposition 2.16: *If G is a connected r-regular graph, then* $\gamma_{hn}(G) \le \left\lceil \frac{n}{2} \right\rceil$.

Proof: A graph G can be classified into two classes depending on the value r where the graph is r-regular as follows:

Case 1: If $r = 1$, then the graph is complete of order two (K_2), so $\gamma_{hn}(\mathrm{G}) = 1$.

Case 2: If $r \ge 2$, then the minimum domination number of G in this case is two, meaning that the graph is a cycle and by Proposition 2.7 $\gamma_{hn}(\mathrm{G}) = \left\lceil \frac{n}{2} \right\rceil$. Otherwise, for every r-regular graph such that $r > 2$, the hn-domination number is less than $\left\lceil \frac{n}{2} \right\rceil$, since the neighborhoods of any vertex is greater than two. Thus, $\gamma_{hn}(\mathrm{G}) \le \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.17: *The hn-domination of graphs* ($\overline{C_n}$, $\overline{W_n}, \overline{S_n}, \overline{P_n}, \overline{K_{n,m}}$, *and* $\overline{K_n}$) is

1) $\gamma_{hn}(\overline{C_n}) = \begin{Bmatrix} 3, & if\ n = 3, 5 \\ 2, if\ n = 4\ or\ n \ge 6 \end{Bmatrix}$.
2) $\gamma_{hn}(\overline{W_n}) = \begin{Bmatrix} 4, & if\ n = 3, 5 \\ 3, if\ n = 4\ or\ n \ge 6 \end{Bmatrix}$
3) $\gamma_{hn}(\overline{S_n}) = \gamma_{hn}(\overline{P_n}) = \gamma_{hn}\left(\overline{K_{n,m}}\right) = 2\ \forall n \ge 2$.
4) $\gamma_{hn}(\overline{K_n}) = n$

Proof:

1) If $G \cong C_n$, then there are three cases as follows:

i) If $n = 3$, then $\overline{C_3} \cong N_3$, thus $\gamma_{hn}(\overline{C_3}) = \gamma_{hn}(N_3) = 3$.

ii) If $n = 5$, then C_5 is self-complementary which means $\overline{C_5} \cong C_5$, thus by Proposition 2.9, $\gamma_{hn}(\overline{C_5}) = \gamma_{hn}(C_5) = 3$.

iii) If $n > 3; n \ne 5$, then there are two ways to calculate the hn-domination number of complement of this cycle.

The first way when $n = 4$, then $\overline{C_4} \cong K_2 \cup K_2$, so by proposition 2.15 $\gamma_{hn}(\overline{C_4}) = 2$. The second way when $n > 5$, then we choose two vertices in $\overline{C_n}$ say u and v such that $d(u, v) = 3$. Thus, in $\overline{C_n}$ the vertex u is adjacent to all vertices in $\overline{C_n}$ except two vertices which are adjacent to it in C_n. Also, the vertex v is adjacent to these two vertices. Therefore, u and v belong to $hn-DS$ in $\overline{C_n}$. Thus, in this case $\gamma_{hn}(\overline{C_n}) = 2$.

2) Since $\overline{W_n} \cong \overline{C_n} \cup K_1$, then by the same procedure in (1) and observation 2.10 $\gamma_{hn}(\overline{W_n}) = \gamma_{hn}(\overline{C_n}) + 1$.

3) a) If $G \cong K_{n,m}$, then the graph $\bar{G}$ contains two components; one of them is a complete graph of order n and the other is a complete graph of order m. Thus, by using observation 2.10 $\gamma_{hn}\left(\overline{K_{n,m}}\right) = 2$. The star graph is isomorphic to complete bipartite graph $K_{n-1,1}$. Therefore, $\gamma_{hn}(\overline{S_n}) = 2$.

b) If $G \cong P_n$, it is easy to check that $\gamma_{hn}(\overline{P_2}) = \gamma_{hn}(\overline{P_3}) = 2$. Now, there are three cases depending on the order of path as follows:

i) If $n = 4$, then P_4 is self complementary, then $\gamma_{hn}(\overline{P_4}) = \gamma_{hn}(P_4) = 2$.

ii) If $n = 5$, then the pendent vertices u and v become the two vertices which are dominating all vertices in $\overline{P_5}$. Thus, $\gamma_{hn}(\overline{P_5}) = 2$.

iii)If $n \ge 6$, then by the same manner in 1(iii), we get the result.

4) It is obvious. □

Theorem 2.18. Let G be a graph has hn-domination number γ_{hn} ,"then in $G - v, v \in D$ if v is adjacent to at least two of the independent vertices"in $V - D$ such that there is no vertex in D dominated these vertices, then $\gamma_{hn}(G - v) \ge \gamma_{hn}(G)$. Otherwise, $\gamma_{hn}(G - v) \le \gamma_{hn}(G)$.

Proof: "Let D be $hn - MD$ with minimum cardinality of the graph G, then there are two cases as follows:

Case 1: If we delete a vertex v, where $v \in D$ then four cases are obtained as follows:

i) *if v is adjacent to at least two of the independent vertices"in $V - D$ such that there is no vertex in D that dominate on these vertices*, then these vertices must belong to $D - v$. Thus, $\gamma_{hn}(G - v) > \gamma_{hn}(G)$.(for example, see Fig.2d).

ii) If v is isolated in G, then $\gamma_{hn}(G - v) < \gamma_{hn}(G)$.

iii) If v is isolated in D and the neighborhoods of v in $V - D$ are dominated by some vertices in the set D, then $\gamma_{hn}(G - v) < \gamma_{hn}(G)$. (as an example, see Fig.2b).

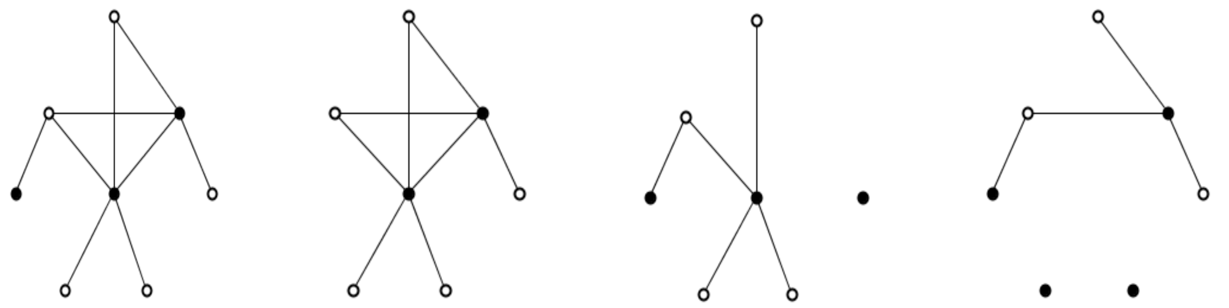
iv) If v is the only vertex adjacent to k vertices in $V - D$ and there is a vertex from the k vertices that dominates the other vertices, then in these cases $\gamma_{hn}(G - v) = \gamma_{hn}(G)$(for example , see Fig.2c, $k = 1$).

Case 2: If we delete a vertex v from $V - D$, then there are three cases as follows:

i) If $u \in D$ is adjacent to v such that the neighborhoods of u in $V - D$ are dominated by other vertex in D and u is not isolated in D,then $\gamma_{hn}(G - v) < \gamma_{hn}(G)$ (for example, see Fig. 3b).

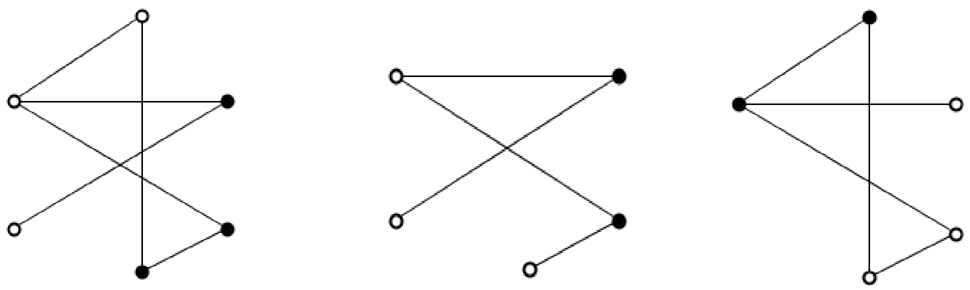
ii) If $|D| = |V - D|$ and $(V - D) - \{v\}$ has $hn - DS$, then $\gamma_{hn}(G - v) < \gamma_{hn}(G)$.(as example, see Fig.3c. Otherwise,

$\gamma_{hn}(G - v) = \gamma_{hn}(G)$ □



(a) Graph G (b) $\gamma_{hn}(G-v)<\gamma_{hn}(G)$ (c)$\gamma_{hn}(G-v)=\gamma_{hn}(G)$ (d)$\gamma_{hn}(G-v)>\gamma_{hn}(G)$

Figure 2. Hn-domination number of a graph $G-v$ when deletion a vertex from D.



(a) Graph G (b)$\gamma_{hn}(G-v)<\gamma_{hn}(G)$ (c)$\gamma_{hn}(G-v)<\gamma_{hn}(G)$

Figure 3. Hn-domination number of a graph $G-v$ when deletion a vertex from $V-D$

Theorem 2.19: *If G has γ_{hn}-set, then $\gamma_{hn}(G-e)\geq\gamma_{hn}(G)$.*

Proof: If G has a γ_{hn}-set of G say . By deleting an edge e from a graph G,we get the following three cases as follows:

Case1: If e is an edge that is incident on two vertices in $V-D$, then the hn-domination is not influenced by this deletion. Thus, $\gamma_{hn}(G-e)=\gamma_{hn}(G)$. (as an example, see Fig.4b).

Case 2: If e is an edge that is incident on two vertices in D, then there are two cases as follows:

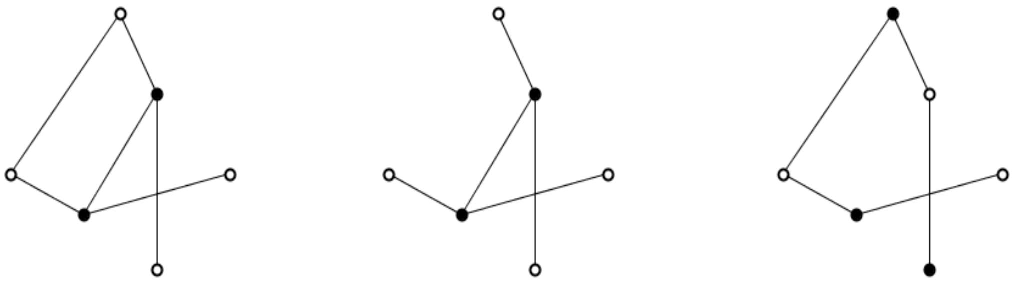
i) If these two vertices are adjacent to at least two independent vertices in $V-D$, then the hn-domination is not influenced by this deletion which means $\gamma_{hn}(G-e)=\gamma_{hn}(G)$.

ii) If these two vertices are adjacent to exactly two adjacent vertices in $V-D$ and these vertices are not adjacent to other adjacent vertices in D, then D has no $hn-DS$. Thus, $\gamma_{hn}(G-e)\geq\gamma_{hn}(G)$.(as an example, see Fig.4c).

Case 3: If e is an edge that is incident on two vertices. One of them in D say v and the other in $V-D$ say u, then there are two cases as follows:

i) If there is another vertex in D which is hn-dominates the vertex u other than v, then $\gamma_{hn}(G-e)=\gamma_{hn}(G)$.

ii) If u is the unique vertex which hn-dominates the vertex v, then D loses the hn-domination . Thus, $\gamma_{hn}(G-e)\geq\gamma_{hn}(G)$. □



(a) Graph G (b)$\gamma_{hn}(G-e)=\gamma_{hn}(G)$ (c)$\gamma_{hn}(G-e)>\gamma_{hn}(G)$

Figure 4. Deletion an edge e that incident two vertices in D or in $V-D$

Theorem 2.20: *If a graph G has γ_{hn} –set, then if e is an edge that is incident on two vertices which are hn-dominated by distinct and independent vertices, then $\gamma_{hn}(G+e) \geq \gamma_{hn}(G)$. Otherwise, $\gamma_{hn}(G+e) \leq \gamma_{hn}(G)$, $e \in \bar{G}$*

Proof: Suppose that D be γ_{hn}-set of a graph G. By adding an edge $e = uv$ $(u, v \in G)$,we get the following three cases as follows:

Case 1: If e is an edge that is incident on two vertices in D, then there are two cases as follows:

i) If v is an isolated vertex in G (as example, see Fig.5b) or all neighborhoods of v *or* u (say v) are in $V - D$ such that they are $hn - DS$ by the other vertices in D, then $D - \{v\}$ is an $hn - DS$ of G. Thus, $\gamma_{hn}(\mathrm{G+e}) < \gamma_{hn}(\mathrm{G})$. (as example, see Fig.5c).

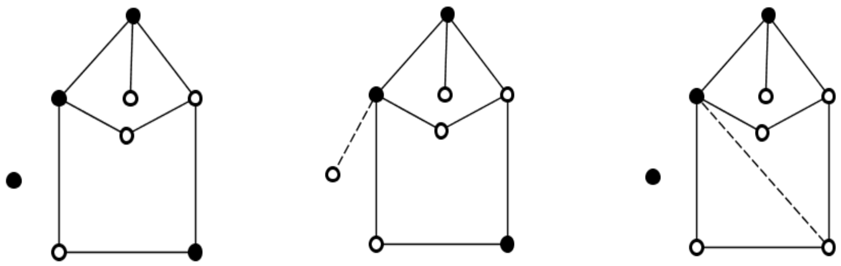
ii) If there is a vertex that belongs to the neighborhood of the vertex v and is not hn-dominated by the other vertex in D, then this addition does not affect $hn - DS$. Therefore, $\gamma_{hn}(\mathrm{G+e}) = \gamma_{hn}(\mathrm{G})$.

Case 2: If e is an edge that is incident on two vertices in $V - D$, then there are two cases as follows:

i) If u and v are hn-dominated by the same vertex or by adjacent vertices, then this addition does not affect $hn - DS$. Therefore, $\gamma_{hn}(\mathrm{G+e}) = \gamma_{hn}(\mathrm{G})$.

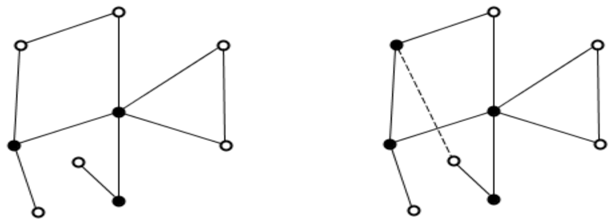
ii) If u and v are not hn-dominated by the same vertex and the vertices in D which are $hn - DS$ the vertices u and v are independent, then D loses the $hn - DS$. Therefore, $\gamma_{hn}(G+e) \geq \gamma_{hn}(G)$. (as example, see Fig.6b).

Case 3: If e is an edge that is incident to two vertices one of them in $V - D$ and the other in D say v, then, if v is adjacent to a vertex in D and u is a pendent vertex in G, then $\gamma_{hn}(\mathrm{G+e}) < \gamma_{hn}(\mathrm{G})$. Otherwise, $\gamma_{hn}(\mathrm{G+e}) = \gamma_{hn}(\mathrm{G})$. □



(a) Graph G (b)$\gamma_{hn}(G+e) < \gamma_{hn}(G)$ (c)$\gamma_{hn}(G+e) = \gamma_{hn}(G)$

Figure 5. Adding an edge e for two vertices in D.



(a) Graph G (b)$\gamma_{hn}(G+e) > \gamma_{hn}(G)$

Figure 6. Adding an edge for two vertices in $V - D$

Conclusion:

In this paper, we introduced a new definition for domination number in graphs, namely hn-domination. The hn-dominating set and hn-domination number for some graphs are found and proved. Also, some operations in hn-domination number are stated and proved. Through this paper, we conclude some properties of hn-domination number.

Conflicts of Interest: None.

References:

1. Harary F. Graph theory. London: Addison-Wesley. 1969.
2. Rad NJ. Some notes on the isolate domination in graphs. AKCE Internat Journal of Graphs Combin. 2017 Aug 1;14(2):112-117.
3. Alikhani S, Fatehi D, Klavžar S. On the structure of dominating graphs. Graphs Combin. 2017 Jul 1;33(4):665-672.
4. Haynes TW, Hedetniemi S, Slater P. Fundamentals of domination in graphs. CRC Press. 1998 Jan 5.
5. Berge C. The theory of graphs its A. London: Methuen Co. 1962.

6. Cockayne EJ, Hedetniemi ST. Towards a theory of domination in graphs. Networks. 1977 Sep 1;7(3):247-261.
7. Al-harere MA, Athraa TB. Inverse bi-domination in graphs. Int J Pure Appl Math. Accepted. 2017 Jun 28.
8. Omran AA, Yasen R. Some properties of Frame domination in graphs. JEAS. 2017; 12(10 SI): 8882-8885.
9. Omran AA, Ibrahim TA. Whole domination in graph. Submitted to Int J Pure Appl Math. 2017.
10. Sridharan N, Subramanian V, Elias M. Bounds on the distance two-domination number of a graph. Graphs Combin. 2017 Oct;18(3): 667-675.

رقم الهيمنة hn في البيان

احمد عبدعلي عمران **حنين حامد عودة**

قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة بابل، بابل، العراق.

الخلاصة:

الهدف من هذا البحث هو تقديم تعريف جديد لرقم الهيمنة في البيان ويسمى رقم الهيمنة hn ويرمز له بالرمز $\gamma_{hn}(G)$. يعرض هذا البحث مفهوم الاتصال والاستقلالية لرقم الهيمنة الجديد ، بالإضافة الى ذلك وضع قيود لهذا الرقم من خلال بعض الخصائص. كذلك تمت دراسة تأثر هذا العدد عند حذف رأس او حذف او اضافة حافة بعمق في هذا البحث.

الكلمات المفتاحية: البيان، رقم الهيمنة hn، عدد الهيمنة hn.