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Faber Polynomial Coefficient Estimates for Subclass of Analytic Bi-Bazilevic Functions Defined by Differential Operator

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Abstract:

In this work, an explicit formula for a class of Bi-Bazilevic univalent functions involving differential operator is given, as well as the determination of upper bounds for the general Taylor-Maclaurin coefficient $N-th\ (N \geq 3)$of a functions belong to this class, are established Faber polynomials are used as a coordinated system to study the geometry of the manifold of coefficients for these functions. Also determining bounds for the first two coefficients of such functions.

In certain cases, our initial estimates improve some of the coefficient bounds and link them to earlier thoughtful results that are published earlier.

Key words: Bi-Bazilevic functions, Faber polynomials, Taylor-Maclaurin coefficients, Taylor-Maclaurin series expansion, Univalent functions.

Introduction:

Let A be the class of all functions $f(z)$ as the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}) \qquad (1.1)$$

which are analytic and normalized in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Also, let S be the subclass of A consist of all functions that are univalent functions in $\mathcal{U}$. A function $f \in S$ has an inverse f^{-1} is defined as follows

$$f^{-1}(f(z)) = z, \quad (z \in \mathcal{U})$$

and

$$f(f^{-1}(w)) = w. \ \left(|w| < r_0(f);\ r_0(f) \geq \tfrac{1}{4}\right) \qquad (1.2)$$

In fact, if $g = f^{-1}$ is the inverse of the function $f \in S$, then g has a Maclaurin series expansion in some disk about the origin which is given by

$$g(w) = f^{-1} = w - a_2 w^2 + (2a_2^2 - a^3)w^3 - (5a_2^3 - 5a_2 a_3)w^4 + \cdots = w + \sum_{n=2}^{\infty} A_n w^n. \qquad (1.3)$$

A function $f \in A$ is called Bi-univalent in $\mathcal{U}$ if both f and its inverse f^{-1} are univalent functions in $\mathcal{U}$. Let $\sum$ denote the family of all Bi-univalent functions in $\mathcal{U}$ which are given by in (1.1).

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A function $f \in A$ is called Bi-Bazilevic if both f and its inverse f^{-1} are Bazilevic in the unite disk $\mathcal{U}$.

From (1.1)

$$(f(z))^{\alpha} = z^{\alpha}(1 + a_2 z^1 + a_3 z^2 + \cdots)^{\alpha}, \quad (0 < \alpha) \ (1.4)$$

Using the binomial expansion for (1.4), to get

$$f(z)^{\alpha} = z^{\alpha}[1 + \alpha(a_2 z + a_3 z^2 + a_4 z^3 + \cdots) + \frac{\alpha(\alpha-1)}{2!}(a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^2 + \cdots]$$
$$= z^{\alpha} + \alpha a_2 z^{\alpha+1} + \alpha a_3 z^{\alpha+2} + \alpha a_4 z^{\alpha+3} + \cdots .$$

Let the class of analytic functions of A_{α} be

$$f(z)^{\alpha} = z^{\alpha} + \sum_{n=2}^{\infty} a_n(\alpha) z^{\alpha+n-1}. \qquad (1.5)$$

For a function $f(z)^{\alpha}$ given by (1.5), define the differential operator $\Gamma_{\beta,\mu}^{m,\lambda}: A_{\alpha} \longrightarrow A_{\alpha}$ as follows:

$$\Gamma_{\beta,\mu}^{0,\lambda} f(z)^{\alpha} = f(z)^{\alpha}$$
$$\Gamma_{\beta,\mu}^{1,\lambda} f(z)^{\alpha} = \left(\frac{1-(\beta-\lambda)}{\mu+\lambda}\right)\left(\Gamma_{\beta,\mu}^{0,\lambda} f(z)^{\alpha}\right) + \frac{\beta-\lambda}{\mu+\lambda} z(\Gamma_{\beta,\mu}^{0,\lambda} f(z)^{\alpha})' \qquad (1.6)$$

$$= \left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^1 z^{\alpha} + \sum_{n=2}^{\infty}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{\mu+\lambda}\right)^1 a_n(\alpha) z^{\alpha+n-1}.$$

$$\Gamma_{\beta,\mu}^{2,\lambda} f(z)^{\alpha} = \left(\frac{1-(\beta-\lambda)}{\mu+\lambda}\right)\left(\Gamma_{\beta,\mu}^{1,\lambda} f(z)^{\alpha}\right) + \frac{\beta-\lambda}{\mu+\lambda} z(\Gamma_{\beta,\mu}^{1,\lambda} f(z)^{\alpha})'$$

$$= \left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^2 z^\alpha + \sum_{n=2}^{\infty}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{\mu+\lambda}\right)^2 a_n(\alpha) z^{\alpha+n-1}.$$

In general,

$$\Gamma_{\beta,\mu}^{m,\lambda} f(z)^\alpha = \Gamma(\Gamma_{\beta,\mu}^{m-1,\lambda} f(z)^\alpha)\left(\frac{1-(\beta-\lambda)}{\mu+\lambda}\right)(\Gamma_{\beta,\mu}^{m-1,\lambda} f(z)^\alpha) + \frac{\beta-\lambda}{\mu+\lambda} z(\Gamma_{\beta,\mu}^{m-1,\lambda} f(z)^\alpha)' \qquad (1.7)$$

$$= \left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m z^\alpha + \sum_{n=2}^{\infty}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{\mu+\lambda}\right)^m a_n(\alpha) z^{\alpha+n-1},$$

where

$(\mu > 0, \beta, \lambda \geq 0\,; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \alpha > 0, z \in \mathcal{U})$.

Remark 1.1 It is easily to see from (1.7), that the operator reduces to several known differential operators by giving specific values to the parameters which have been studied by following earlier authors for instance:

I. For $\alpha = 1, \lambda = 0$ and $\mu = 1$, we get to the operator $D^m f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\beta]^m a_n z^n$,

which was introduced by Al-Obouudi (1).

II. For $\alpha = 1, \lambda = 0$ $\mu = 1$, and $\beta = 1$, we get to the operator $D^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n$, which was introduced by Salagean (2).

Now, using the differential operator $\Gamma_{\beta,\mu}^{m,\lambda} f(z)^\alpha$ to define a new class of Bi-Bazilevic type functions as follows:

Definition 1.2 The function $f(z)^\alpha$ which has the form (1.5) belongs to the class $\mathcal{F}_{\beta,\mu}^{m,\lambda}(\alpha,\zeta,\gamma)$ satisfies the following conditions

$$Re\left\{\frac{\Gamma_{\beta,\mu}^{m,\lambda} f(z)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m z^\alpha}\right\} > \gamma\left|\frac{\Gamma_{\beta,\mu}^{m,\lambda} f(z)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m z^\alpha} - 1\right| + \zeta, \quad (z \in \mathcal{U}) \qquad (1.8)$$

and

$$Re\left\{\frac{\Gamma_{\beta,\mu}^{m,\lambda} g(w)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m w^\alpha}\right\} > \gamma\left|\frac{\Gamma_{\beta,\mu}^{m,\lambda} g(w)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m w^\alpha} - 1\right| + \zeta, \quad (w \in \mathcal{U}) \qquad (1.9)$$

The above conditions are equivalent to

$$Re\left\{\frac{\Gamma_{\beta,\mu}^{m,\lambda} f(z)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m z^\alpha}\right\} > \frac{\zeta-\gamma}{1-\gamma}, (z \in \mathcal{U}) \qquad (1.10)$$

and

$$Re\left\{\frac{\Gamma_{\beta,\mu}^{m,\lambda} g(w)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m w^\alpha}\right\} > \frac{\zeta-\gamma}{1-\gamma}, \quad (w \in \mathcal{U}) \qquad (1.11)$$

respectively for some $(\mu > 0, \beta, \gamma,\ \lambda \geq 0\,; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \alpha > 0, z \in \mathcal{U})$ and $0 \leq \zeta < 1, g = f^{-1}$ is defined by (1.3).

Remark 1.3

I. If take $m = 1, \lambda = \gamma = 0$ and $\mu = \beta = 1$ in Definition 1.1, then the class $\mathcal{F}_{\beta,\mu}^{m,\lambda}(\alpha,\zeta,\gamma)$ reduced to class of Bi-Bazilevic functions of order ζ and type α (see Bazilevic (3)).

$$Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha}\right\} > \zeta, \qquad (z \in \mathcal{U})$$

and

$$Re\left\{g'(z)\left(\frac{w}{g(w)}\right)^{1-\alpha}\right\} > \zeta, \qquad (w \in \mathcal{U}).$$

II. If take $m = 1,\ \lambda = \gamma = 0\ , \alpha = 1$ and $\mu = \beta = 1$ in Definition 1.1 then, we have the class of Bi-univalent function

$$Re\{f'(z)\} > \zeta, \qquad (z \in \mathcal{U})$$

and

$$Re\{g'(z)\} > \zeta, \qquad (w \in \mathcal{U})$$

which was introduced by Srivastava et al. (4).

Estimate on the coefficients bounds of classes of meromorphic and univalent functions were widely researched in the literature. For instance, in 1948 Schiffer (5) proved that the estimate $|a_2| \leq \frac{2}{3}$ for meromorphic and univalent functions f with $a_0 = 0$ and Duren (6) obtained that if $a_1 = a_2 = \cdots = a_k = 0$ for $1 \leq k \leq \frac{n}{2}$, therefor $|a_n| \leq \frac{2}{n+1}$.

Hamidi S. G., Janani T, Murugusundaramoorthy G., and Jahangiri J. M. (7) considered the inverse function $g = f^{-1}$, where $f \in \mathcal{F}_{1,1}^{1,0}(1,\zeta,0)$ and obtained the bound $2(1-\zeta)/(n+1)$ if $(\frac{n-1}{n}) \leq \zeta < 1$. This restriction forced on ζ is a very tight restriction since the class $\mathcal{F}_{1,1}^{1,0}(1,\zeta,0)$ shrinks for large values of n.

The real difficulty emerges when the Bi-univalency conditions are forced on the meromorphic functions f and its inverse $g = f^{-1}$. The sudden and bizarre conduct of the coefficients of meromorphic functions f and their inverse $g = f^{-1}$, prove the investigation of the coefficient bounds for Bi-univalent functions to be extremely challenging.

In order to extend the results of Hamidi S. G., Janani T, Murugusundaramoorthy G., and Jahangiri J. M. (7) to a general class of meromorphic Bi-univalent functions, we use the instrument of the well-known Faber polynomial expansions to determine estimates for a general subclass of analytic Bi-Bazilevic functions. Furthermore, we prove the unperdictability of the early first two cofficients of such Bi-Bazilevic functions that belong to this class which is the best estimate in the literature.

Coefficient Estimates

Consider the function $f \in A$ of the form (1.1). Then the coefficients of its inverse function $g = f^{-1}$, may be represented by Faber polynomial expansion as

$$g(w) = f^{-1}(w)$$
$$= w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^n, \qquad (2.1)$$

where $n-1 \geq 1, -n \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and $K_{n-1}^{-n} = K_{n-1}^{-n}(a_2, a_3, \dots, a_n) = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5}[a_5 + (-n+2)a_3^2] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6}[a_6 + (-2n+5)a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j.$

and V_j with $7 \leq j \leq n$ is a homogeneous polynomial of degree j in the coefficients $|a_2|, |a_3|, \dots, |a_n|$.

In general, for any real number p, an expansion of $K_n^p = K_n^p(a_2, a_3, \dots, a_n)$ (e.g. see [(8), equation (4)]) is given as:.

$$K_{n-1}^p$$
$$= p a_n + \frac{p(p-1)}{2} E_{n-1}^2 + \frac{p!}{(p-3)!\,3!} E_{n-1}^3 + \cdots$$
$$+ \frac{p!}{(p-n+1)!\,(n-1)!} E_{n-1}^{n-1}, \qquad (2.2)$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, \dots, a_n)$, are homogeneous polynomials explicated,

$$E_{n-1}^p(a_2, a_3, \dots, a_n) = \sum_{n=2}^{\infty} \frac{p!\,(a_2)^{\sigma_1} \dots (a_n)^{\sigma_{n-1}}}{\sigma_1! \dots \sigma_{n-1}!},$$
$$for \quad p \leq n$$

such that $a_1 = 1$ and taking the sum on over all non-negative integers $\sigma_1, \dots, \sigma_{n-1}$, satisfing

$$\begin{cases} \sigma_1 + \sigma_2 + \cdots + \sigma_{n-1} = p \\ \sigma_1 + 2\sigma_2 + \cdots + (n-1)\sigma_{n-1} = n-1. \end{cases}$$

It is clear that $E_{n-1}^{n-1}(a_2, a_3, \dots, a_n) = a_2^{n-1}$(9). Evidently, $E_n^n(a_1, a_2, \dots, a_n) = E_1^n$; this means that the first polynomials and last polynomials are

$$E_n^n = a_1^n, E_n^1 = a_n.$$

For instance the first three elements of K_{n-1}^{-n} are:

$K_1^{-2} = -2a_2,$

$K_2^{-3} = 3(2a_2^2 - a_3),$

$K_3^{-4} = -4(5a_2^3 - 5a_2 a_3 + a_4).$

A similar Faber polynomial expansion formula holds for the coefficients of g, the inverse function of f. The Faber polynomials presented which play an important role in different areas of mathematical sciences, particularly in geometric function theory (Schiffer (5)). The recent interest for the calculus of Faber polynomials, particularly when it includes $g = f^{-1}$, the inverse of f (see (10), (11), (12), (13), (14), and (15)), flawlessly fits our case for the Bi-univalent functions. As a result, we can state the following.

Theorem 2.1 For $\mu > 0$; $\beta, \lambda, \gamma \geq 0$, $0 \leq \zeta < 1$; both the functions $f(z)^\alpha$ and its inverse map $g(z)^\alpha = f(z)^{-\alpha}$ are in $\mathcal{F}_{\beta,\mu}^{m,\lambda}(\alpha, \zeta, \gamma)$. If $a_k = 0$; $2 \leq k \leq n-1$, then

$$|a_n| \leq \frac{2(1-\zeta)}{\alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m} \quad \text{for } n \geq 4.$$

Proof: Since the function $f(z)^\alpha \in \mathcal{F}_{\beta,\mu}^{m,\lambda}(\alpha, \zeta, \gamma)$ and its inverse function $g(z)^\alpha = f(z)^{-\alpha} \in \mathcal{F}_{\beta,\mu}^{m,\lambda}(\alpha, \zeta, \gamma)$, then there is a positive real part functions $p(z)$ and $q(z)$ in the unit disk $\mathcal{U}$

$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$,

where $Re\{p(z)\} > 0$ and $Re\{q(z)\} > 0$. Therefore, we get

$$(1-\gamma)\frac{\Gamma_{\beta,\mu}^{m,\lambda} f(z)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m z^\alpha} + \gamma$$
$$= \zeta + (1-\zeta)p(z)$$
$$= 1 + (1-\zeta)\sum_{n=1}^{\infty} c_n z^n, \qquad (2.3)$$

and

$$(1-\gamma)\frac{\Gamma_{\beta,\mu}^{m,\lambda} g(w)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m w^\alpha} + \gamma$$
$$= \zeta + (1-\zeta)q(w)$$
$$= 1 + (1-\zeta)\sum_{n=1}^{\infty} d_n w^n. \qquad (2.4)$$

By application, Faber polynomial expansion to the power series for functions $f(z)^\alpha \in \mathcal{F}_{\beta,\mu}^{m,\lambda}(\alpha, \zeta, \gamma)$. Therefore, the left hand sides of the equations (2.3) and (2.4) can be expressed by

$$(1-\gamma)\frac{\Gamma_{\beta,\mu}^{m,\lambda} f(z)^\alpha}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m z^\alpha} + \gamma$$
$$= 1 + \sum_{n=2}^{\infty} F_{n-1}^\alpha(a_2, a_3, \dots, a_n) z^{n-1}$$

$$= 1 + \sum_{n=2}^{\infty} \alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m K_{n-1}^{\alpha}(a_2, a_3, \dots, a_n)z^{n-1}, \quad (2.5)$$

and the same way, we obtain

$$(1-\gamma)\frac{\Gamma_{\beta,\mu}^{m,\lambda} g(w)^{\alpha}}{\left(\frac{1+(1-\alpha)(\lambda-\beta)}{\mu+\lambda}\right)^m w^{\alpha}} + \gamma = 1 + \sum_{n=2}^{\infty} F_{n-1}^{\alpha}(A_2, A_3, \dots, A_n)w^{n-1}$$

$$= 1 + \sum_{n=2}^{\infty} \alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m K_{n-1}^{\alpha}(A_2, A_3, \dots, A_n)w^{n-1} \quad (2.5)$$

where K_{n-1}^{p} is defined by (2.2) and

$$A_n = \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n), \quad n = 2,3, \dots.$$

Comparing the corresponding coefficient of (2.3) and (2.5), we get

$$\sum_{n=2}^{\infty} \alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m K_{n-1}^{\alpha}(a_2, a_3, \dots, a_n)(1-\zeta)c_{n-1}, \quad (2.7)$$

and similarly, from (2.4) and (2.6), we get

$$\sum_{n=2}^{\infty} \alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m K_{n-1}^{\alpha} \times (A_2, A_3, \dots, A_n)(1-\zeta)d_{n-1}. \quad (2.8)$$

Which under the assumption $a_k = 0$ for $2 \le k \le n-1$, we conclude that

$$\alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m a_n = (1-\zeta)c_{n-1}, \quad (2.9)$$

and

$$-\alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m a_n = (1-\zeta)d_{n-1}. \quad (2.10)$$

Note that, in virtue of Caratheodory Lemma (e.g. (8)), we have

$$|c_n| \le 2 \quad \text{and} \quad |d_n| \le 2, \quad (n \in \mathbb{N}).$$

By taking the absolute value of equalities (2.9) and (2.10), we obtain the required bound

$$|a_n| = \frac{(1-\zeta)|c_{n-1}|}{\alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m} = \frac{(1-\zeta)|d_{n-1}|}{\alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m} \le \frac{2(1-\zeta)}{\alpha(1-\gamma)\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m}.$$

The proof is complete.

Putting $m = 0$ in Theorem 2.1, we obtain the following Corollary.

Corollary 2.1 For $\gamma \ge 0, 0 \le \zeta < 1$; both the functions $f(z)^{\alpha}$ and its inverse map $g(z)^{\alpha} = f(z)^{-\alpha}$ are in $\mathcal{F}_{\beta,\mu}^{0,\lambda}(\alpha, \zeta, \gamma)$. If $a_k = 0$; $2 \le k \le n-1$, then

$$|a_n| \le \frac{2(1-\zeta)}{\alpha(1-\gamma)} \quad \text{for } n \ge 4.$$

Putting $m = 0, \gamma = 0, \zeta = 0$ and $\alpha = 2$ in Theorem 2.1, we gets the following Corollary.

Corollary 2.2 For both the functions $f(z)^{\alpha}$ and its inverse map $g(z)^{\alpha} = f(z)^{-\alpha}$ are in $\mathcal{F}_{\beta,\mu}^{0,\lambda}(2, 0, 0)$. If $a_k = 0$; $2 \le k \le n-1$, then

$$|a_n| \le 1 \quad \text{for } n \ge 4.$$

Putting $m = 1, \lambda = \mu = 0, \gamma = 0$, and $\beta = 1$ in Theorem 2.1, we gets the following Corollary.

Corollary 2.3 For $0 \le \zeta < 1$; both the functions $f(z)^{\alpha}$ and its inverse map $g(z)^{\alpha} = f(z)^{-\alpha}$ are in $\mathcal{F}_{1,0}^{1,0}(\alpha, \zeta, 0)$. If $a_k = 0$; $2 \le k \le n-1$, then

$$|a_n| \le \frac{2(1-\zeta)}{(n-1)+\alpha} \quad \text{for } n \ge 4.$$

The above Corollary 2.3 reduces to the result ((16), Theorem 2.1) for analytic Bi-Bazilevic functions of order α and type $\zeta; 0 \le \zeta < 1$which is studied by Jay M. Jahangiri and Samaneh G. Hamidi.

Putting $m = 1, \lambda = 0, \alpha = 1$ and $\gamma = 0$ in Theorem 2.1, we obtain the following:

Corollary 2.4 For> 0; $\beta \ge 0$, $0 \le \zeta < 1$; both functions $f(z)^{\alpha}$ and its inverse map $g(z)^{\alpha} = f(z)^{-\alpha}$ are in $\mathcal{F}_{\beta,\mu}^{1,0}(1, \zeta, 0)$. If $a_k = 0$; $2 \le k \le n-1$, then

$$|a_n| \le \frac{2(1-\zeta)}{1+\beta(n-1)} \quad \text{for } n \ge 4.$$

Note that Corollary 2.4 reduces to the result ((17), Theorem 1) for analytic Bi-univalent functions of order α and type ζ ; $0 \le \zeta < 1$ which is studied by Jay M. Jahangiri and Samaneh G. Hamidi.

Theorem 2.2 For $\mu > 0$; $\beta, \lambda, \gamma \ge 0$ and $0 \le \zeta < 1$. If both functions $f(z)^{\alpha}$ and its inverse map $g(z)^{\alpha} = f(z)^{-\alpha}$ are in $\mathcal{F}_{\beta,\mu}^{m,\lambda}(\alpha, \zeta, \gamma)$, then one has the following:

$$|a_2| \le \begin{cases} \sqrt{\frac{4(1-\zeta)}{\alpha(\alpha+1)(1-\gamma)\psi_2}}, & 0 \le \zeta < 1 - \frac{(1-\gamma)\psi_1^2}{\alpha(\alpha+1)\psi_2}, \\ \frac{2(1-\zeta)}{\alpha(1-\gamma)\psi_1}, & 1 - \frac{(1-\gamma)\psi_1^2}{\alpha(\alpha+1)\psi_2} \le \zeta < 1, \end{cases}$$

where

$$\psi_1 = \left(\frac{1+\alpha(\beta-\lambda)}{1+(1-\alpha)(\lambda-\beta)}\right)^m, \quad \psi_2 = \left(\frac{1+(\alpha+1)(\beta-\lambda)}{1+(1-\alpha)(\lambda-\beta)}\right)^m,$$

and

$$|a_3 - a_2^2| = \frac{(1-\zeta)}{2\alpha(1-\gamma)\psi_2}(|c_2| + |d_2|) \le \frac{2(1-\zeta)}{\alpha(1-\gamma)\psi_2}.$$

Proof If putting $n = 2$ and $n = 3$ in (2.7) and (2.8), respectively, we get

$$\alpha(1-\gamma)\psi_1 a_2 = (1-\zeta)c_1, \quad (2.11)$$

$$\frac{\alpha(\alpha-1)(1-\gamma)}{2!}\psi_2 a_2^2 + \alpha(1-\gamma)\psi_2 a_3 = (1-\zeta)c_2. \quad (2.12)$$

$$\alpha(1-\gamma)\psi_1 A_2 = (1-\zeta)d_1, \quad (2.13)$$

and

$$\frac{\alpha(\alpha-1)(1-\gamma)}{2!}\psi_2 A_2^2 + \alpha(1-\gamma)\psi_2 A_3 = (1-\zeta)d_2, \quad (2.14)$$

and for suitable values of A_2 and A_3, we have

$$-\alpha(1-\gamma)\psi_1 a_2 = (1-\zeta)d_1, \quad (2.15)$$

and

$$\frac{\alpha(\alpha-1)(1-\gamma)}{2!}\psi_2 a_2^2 + \alpha(1-\gamma)(2a_2^2 - a_3)\psi_2 = (1-\zeta)d_2. \quad (2.16)$$

From (2.11) and (2.15), we conclude by the Caratheodory lemma

$$|a_2| = \frac{(1-\zeta)|c_1|}{\alpha(1-\gamma)\psi_1} = \frac{(1-\zeta)|d_1|}{\alpha(1-\gamma)\psi_1} \le \frac{2(1-\zeta)}{\alpha(1-\gamma)\psi_1}.$$

On the other side, by adding the equations (2.12) and (2.16), we get

$$\alpha(\alpha+1)(1-\gamma)\psi_2 a_2^2 = (1-\zeta)(c_2+d_2). \quad (2.17)$$

Solving the equation (2.17) and applying the Caratheodory lemma, we obtain

$$|a_2| \le \sqrt{\frac{4(1-\zeta)}{\alpha(\alpha+1)(1-\gamma)\psi_2}}.$$

Subtracting (2.16) from (2.12), we deduce

$$2\alpha(1-\gamma)(a_3 - a_2^2)\psi_2 = (1-\zeta)(c_2-d_2). \quad (2.18)$$

Dividing by $2\alpha(1-\gamma)\psi_2$ and by applying the Caratheodory lemma. Consequently, we have

$$|a_3 - a_2^2| = \frac{(1-\zeta)}{2\alpha(1-\gamma)\psi_2}(|c_2| + |d_2|) \le \frac{2(1-\zeta)}{\alpha(1-\gamma)\psi_2}.$$

This evidently completes the proof for theorem.

Putting $m = 0$ in Theorem 2.2, to get the following Corollary:

Corollary 2.5 For $\mu > 0$; $\beta, \lambda, \gamma \ge 0$ and $0 \le \zeta < 1$. If both functions $f(z)^\alpha$ and its inverse map $g(z)^\alpha = f(z)^{-\alpha}$ are in $\mathcal{F}_{\beta,\mu}^{0,\lambda}(\alpha,\zeta,\gamma)$, then one has the following:

$$|a_2| \le \begin{cases} \sqrt{\frac{4(1-\zeta)}{\alpha(\alpha+1)(1-\gamma)}}, & 0 \le \zeta < 1 - \frac{(1-\gamma)}{\alpha(\alpha+1)} \\ \frac{2(1-\zeta)}{\alpha(1-\gamma)}, & 1 - \frac{(1-\gamma)}{\alpha(\alpha+1)} \le \zeta < 1, \end{cases}$$

and

$$|a_3 - a_2^2| \le \frac{2(1-\zeta)}{\alpha(1-\gamma)}.$$

Putting $\gamma = \lambda = 0$ and $\beta = m = 1$ in Theorem 2.2, to get Theorem 2.2 in (16) which is studied by Jay M. Jahangiri and Samaneh G. Hamidi.

Corollary 2.6 If $0 \le \zeta < 1$; both functions $f(z)^\alpha$ and its inverse map $g(z)^\alpha = f(z)^{-\alpha}$ are in $\mathcal{F}_{1,\mu}^{1,0}(\alpha,\zeta,0)$, then one has the following:

$$|a_2| \le \begin{cases} \sqrt{\frac{4(1-\zeta)}{(1+\alpha)(2+\alpha)}}, & 0 \le \zeta < \frac{1}{2+\alpha} \\ \frac{2(1-\zeta)}{1+\alpha}, & \frac{1}{2+\alpha} \le \zeta < 1, \end{cases}$$

and

$$|a_3 - a_2^2| \le \frac{2(1-\zeta)}{2+\alpha}.$$

Putting $\alpha = 0$ in Corollary 2.6, reduces to the result [(16), Corollary 2.1] for analytic Bi-starlike functions of order ζ ; $0 \le \zeta < 1$ which is studied by Jay M. Jahangiri and Samaneh G. Hamidi.

Corollary 2.7 If $0 \le \zeta < 1$; both functions $f(z)^\alpha$ and its inverse map $g(z)^\alpha = f(z)^{-\alpha}$ are in $\mathcal{F}_{0,1}^{1,0}(\alpha,\zeta,0)$, then one has the following:

$$|a_2| \le \begin{cases} \sqrt{2(1-\zeta)}, & 0 \le \zeta < \frac{1}{2} \\ 2(1-\zeta), & \frac{1}{2} \le \zeta < 1, \end{cases}$$

and

$$|a_3 - a_2^2| \le 1-\zeta.$$

Putting $\lambda = \gamma = 0$, and $m = \alpha = 1$ in Theorem 2.2, reduces to the result ((17), Theorem 2).

Corollary 2.8 If $\mu > 0, \beta \ge 0$, $0 \le \zeta < 1$; both functions $f(z)^\alpha$ and its inverse map $g(z)^\alpha = f(z)^{-\alpha}$ are in $\mathcal{F}_{\beta,\mu}^{1,0}(1,\zeta,0)$, then one has the following:

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\zeta)}{1+2\beta}}, & 0 \le \zeta < \frac{1+2\beta-\beta^2}{2(1+2\beta)} \\ \frac{2(1-\zeta)}{1+\beta}, & \frac{1+2\beta-\beta^2}{2(1+2\beta)} \le \zeta < 1, \end{cases}$$

and

$$|a_3 - a_2^2| \le \frac{2(1-\zeta)}{2+\beta}.$$

Note that the above Corollary 2.8 is an improvement of the estimates introduced by Frasin and Aouf [(18), Theorem 3.2].

Putting $\beta = 1$ in Corollary 2.8, reduces to the result [(19), Corollary 7] which is studied by Serap Bulut.

Corollary 2.9 If $\mu > 0$, $0 \le \zeta < 1$; both functions $f(z)^\alpha$ and its inverse map $g(z)^\alpha = f(z)^{-\alpha}$ are in $\mathcal{F}_{1,\mu}^{1,0}(1,\zeta,0)$, then one has the following:

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\zeta)}{3}}, & 0 \le \zeta < \frac{1}{3} \\ (1-\zeta), & \frac{1}{3} \le \zeta < 1, \end{cases}$$

and

$$|a_3 - a_2^2| \le \frac{2(1-\zeta)}{3}.$$

Conclusion:

In this work, we have presented new method for class of Non-Bi-Bazilevic functions by used Faber polynomials and determination of upper bounds for functions belong to this class where we noticed our initial estimates improve some of the coefficient bounds.

Conflicts of Interest: None.

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تقـديرات معامل متعـددة حدود فيبر على فئة فرعيه من دوال باي بازا ليفج التحليلية المعرفة بواسطـة المؤثر التفــاضلي

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الخلاصـة:

في هذا العمل ،تم إعطاء صيغة صريحة لفئة من دوال ثنائي بازا ليفج احادية التكافؤ التي تتضمن مؤثر تفاضلي معين، وكذلك تحديد الحدود العليا لمعامل تايلور-ما كلورين العام $N-th\ (N\ \geq 3)$ لدوال تنتمي إلى هذه الفئة، استخدمت مصطلحات متعددة الحدود من فيبر كنظام منسق لدراسة الشكل الهندسي للعديد من المعاملات لهذه الدوال. كذألك تحديد قيود المعاملات الأولية لهذه الدوال.

في حالات معينه، تعمل تقديراتنا الأولية على تحسين بعض حدود المعامِلات وربطها بالنتائج المدروسة السابقة التي تم نشرها في وقت سابق.

الكلمات المفتاحية: دوال باي بازا ليفج، متعددة حدود فيبر، معاملات تايلور- ما كلورين، توسيع سلسله تايلور- ما كلورين، الدوال احادية التكافؤ.