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Solvability of Some Types for Multi-fractional Integro-Partial Differential Equation

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Abstract:

In this article, the solvability of some proposal types of the multi-fractional integro-partial differential system has been discussed in details by using the concept of abstract Cauchy problem and certain semigroup operators and some necessary and sufficient conditions.

Keywords: Fixed point theorem, Fractional calculus, Mild solution, Nonlinear functional analysis, Semigroup theorems.

Introduction:

There are many types of integro-differential equations which are referred to physical models, engineering models and other fields of applications. Many authors are trying to develop some of the methods to solve them; therefore the authors have discussed the existence solution of the integro-differential equation such as:

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}-\left(\beta_1+\beta_2\int_0^L\left|\frac{\partial u(y,s)}{\partial y}\right|^2 dy\right)\frac{\partial^2 u}{\partial x^2}+g\left(\frac{\partial u}{\partial t}\right)=0$$

where $\beta_1,\beta_2,L>0$, g is a nondecreasing numerical function. This equation can be converted to the abstract differential equation as follows:

$$u''+B^2u+M\left(\left\|B^{1/2}u\right\|^2\right)Bu+g(u')=0$$

The study of the existence of mild solution for an integro-differential equation description is as follows:

$$\frac{\partial^2 u}{\partial t^2}-\lambda\frac{\partial^4 u}{\partial t^2\partial x^2}+\frac{\partial^4 u}{\partial x^4}-\left(\beta_1+\beta_2\int_0^L\left|\frac{\partial u(y,s)}{\partial y}\right|^2 dy\right)\frac{\partial^2 u}{\partial x^2}=f\left(\frac{\partial u}{\partial t}\right)$$

which transformed into the abstract differential equation as follows:

$$Ku''(t)=A\,u(t)+f[t,\ u(t),\ u'(t)]\quad ,$$

see (4) and the references therein.

Recently, the fractional order differential equation model is more interesting and more descriptive to be applied in various branches such as engineering and science. So, the abstract Cauchy problems with fractional order derivative simulated to some problems of the fractional differential equation have appeared in many articles for many applications in real life (1-19).

Consider the following fractional order integro-partial differential equations:

$$\begin{cases} {}^RD_t^{\alpha_1}\,{}^RD_t^{\alpha_2}z(t,x)-\beta\,{}^RD_t^{\alpha_1}\,{}^RD_t^{\alpha_2}\,{}^RD_x^{\alpha}z(t,x)+{}^RD_x^{2\alpha}z(t,x) \\ \quad -\left(\omega_1+\omega_2\int_0^L\left|\frac{\partial z(y,s)}{\partial y}\right|^2 dy\right){}^RD_x^{\alpha}z \\ \quad =f\left[t,z(t,x),{}^RD_t^{\gamma_1}z(t,x)\right]g\left[t,z(t,x),I_t^{\gamma_2}z(t,x)\right] \qquad (1.1) \\ {}^RD_t^{\alpha_1+\alpha_2-1}\,z(t,x)\Big|_{t=0}=\tilde{z}_0 \\ {}^RD_t^{\alpha_1+\alpha_2-2}\,z(t,x)\Big|_{t=0}=\tilde{y}_0 \\ {}^RD_t^{\alpha_2-1}\,z(t,x)\Big|_{t=0}=\tilde{v}_0 \end{cases}$$

where $\omega_1,\omega_2,L>0$, $z(t,x)$ is denoted the deviation x at t, $1<\alpha\le 2$, $1<\alpha_1+\alpha_2\le 2$ and $0<\gamma_1,\gamma_2\le 1$, $\beta\ge 0$, and ${}^RD_t^{\nu}$ is a Riemann-Liouville fractional derivative of order $\nu>0$, $x\in X$ (X is a Banach space). Now, let A^α be an operator defined as $A^\alpha z(t,x)={}^RD_x^{\alpha}z(t,x)$, so

$$(I-\beta A^\alpha)\,{}^RD_t^{\alpha_1}\,{}^RD_t^{\alpha_2}z(t,x)+A^{2\alpha}z(t,x)-\left(\omega_1+\omega_2M\left[\left\|A^{\alpha/2}z\right\|^2\right]\right)A^\alpha z(t,x)=f\left[t,z(t,x),{}^RD_t^{\gamma_1}z(t,x)\right]g\left[t,z(t,x),I_t^{\gamma_2}z(t,x)\right]. \qquad (1.2)$$

such that M is a real function defined on $\mathbb{R}^+$. So, the equation (1.2) written in abstract Cauchy form as follows;

$$\begin{cases} P^R D_t^{\alpha_1 R} D_t^{\alpha_2} z(t,x) = \tilde{A}^\alpha z(t,x) \\ \quad +f\left[t, z(t,x), {}^R D_t^{\gamma_1} z(t,x)\right] \\ \quad g\left[t, z(t,x), I_t^{\gamma_2} z(t,x)\right] \\ {}^R D_t^{\alpha_1+\alpha_2-1} z(t,x)\big|_{t=0} = \tilde{z}_0 \\ {}^R D_t^{\alpha_1+\alpha_2-2} z(t,x)\big|_{t=0} = \tilde{y}_0 \\ {}^R D_t^{\alpha_2-1} z(t,x)\big|_{t=0} = \tilde{v}_0 \end{cases} \qquad (1.3)$$

where $\tilde{A}^\alpha = \left(\omega_1 + \omega_2 M\left[\|A^{\alpha/2} z\|^2\right]\right) A^\alpha - A^{2\alpha}$ and $P = (I - \beta A^\alpha)$ are linear operators on a Banach space

$C_\nu^{RL}([0,T], X) = \{z \in C([0,T]) : {}^R D_t^\nu z \in C([0,T])\}$

with norm $\|z\|^* = \|z\|_C + \|{}^R D_t^\nu z\|_C$, ($\|.\|_C$ is the sup norm in $C([0,T])$), $\nu = max\{\alpha, \alpha_1 + \alpha_2, \gamma_1\}$ and the nonlinear functions

$f: J \times C_\nu^{RL}(J,X) \times C_\nu^{RL}(J,X) \to C_\nu^{RL}(J,X)$ and $g: J \times X \times X \to X$, $(J = [0,T])$.

The scientific problem of the fractional order integro-partial differential equations cannot be solved in an analytic or approximated way even sometimes they are difficult to be studied. Also, their behaviors for their solutions are not appearing in general thus it needs more effort and practice, so these problems have taken our interest.

The aim of this article is to study and present all the results of the solvability for the proposal classical fractional order integro-partial differential equations such that some particular types are cases of this proposal problem. They have an approach which is using a nonlinear functional analysis theorems and abstract Cauchy problem and involving semigroup operators with Mainardi's function category as well as a fixed-point theorem as a basic theory for appearing the solvability with some resent sufficient and necessary conditions for this issue.

Preliminaries

The following notations, definitions, assumptions and results needed later on.

Definition (1), (15):

The fractional integration with order $\alpha > 0$ is

$I^\alpha h(s) = \frac{1}{\Gamma(\alpha)} \int_0^s \frac{h(t)}{(s-t)^{1-\alpha}} dt, \; s > 0, \; \alpha > 0$

Where $\Gamma(u) = \int_0^\infty e^{-s} s^{u-1} ds, \; u > 0$.

Definition (2), (15):

The fractional derivational with order $\alpha > 0$, (Riemann – Liouvill) is

$${}^R D^\alpha h(s) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{ds^n} \int_0^s \frac{h(t)}{(s-t)^{\alpha+1-n}} dt,$$
$$s > 0, \; n-1 < \alpha \le n$$

Definition (3), (15):

The function $M_\alpha(u) = \sum_{L=0}^\infty \frac{(-1)^L u^L}{L!\,\Gamma(-\alpha(L+1)+1)}$,

Where $0 < \alpha < 1$, $u \in \not{C}$

is called the Mainardi's function

Remark (1), (12):

It's clear that, The Mainardi's function satisfies the following:

a. $\int_0^\infty M_\alpha(u) du = 1$

b. $\int_0^\infty u^n M_\alpha(u) dr = \frac{\Gamma(n+1)}{\Gamma(\alpha n+1)}, \quad n \in N^+$.

c. For any $\lambda \in \not{C}$ and $0 < \beta < 1$ then :

$e^{-\lambda^\beta} = \int_0^\infty \frac{\beta}{r^{\beta+1}} M_\beta(r^{-\beta}) e^{-\lambda r} dr$

Definition (4), (7):

A one parameter family $\{S(s), s \ge 0\}$ is called a semigroup on a Banach space X if

i. $S(0) = I$

ii. $S(s+t) = S(s)\,S(t)$, for every $t, s \ge 0$.

Definition (5) ,(14):

A semigroup $S(s), s \ge 0$ is uniformly continuous if $lim_{s\to 0} \|S(s) - I\| = 0$

Definition (6), (14):

A semigroup $S(s), s \ge 0$ is a strongly continuous if for each $x \in X$ then

$lim_{s\to 0} S(s)x = x$.

Definition (7), (14):

The operator A which defined by:

$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \frac{dT(t)x}{dt}\Big|_{t=0}$, for $x \in D(A)$

is a linear operator and generated a semigroup $T(t), t \ge 0$ where

$D(A) = \left\{x \in X, \; \lim_{t\to 0} \frac{T(t)x - x}{t} \;\text{ exists}\right\}$.

Remark (2), (14):

If $S(s)$, $s \ge 0$ strongly continuous semigroup generated by B then,

a) For $y \in X$; $lim_{g\to 0} \frac{1}{g} \int_t^{t+g} S(s) y ds = S(t)y$.

b) for $y \in X$; $\int_0^t S(s) y ds \in D(B)$ and $B \int_0^t S(s) y ds = S(t)y - y$

c) for $y \in D(B)$ if $S(s)y \in D(B)$ then $\frac{d}{ds} S(s)y = B\,S(s)y = S(s)By$

for $\lambda > 0$ and $y \in X$, then

$R(\lambda; B) = \int_0^\infty e^{-\lambda s} S(s) ds$.

Remark (3), (15):

${}_0^R D_t^\alpha \left({}_0^R D_t^\beta f(t) \right) =$

${}_0^R D_t^{\alpha+\beta} f(t) - \sum_{j=1}^m [{}_0^R D_t^{\beta-j} f(t)]_{t=0} \frac{t^{-\alpha-j}}{\Gamma(1-\alpha-j)}$

where $m-1 \le \beta < m$, $n-1 \le \alpha < n$, $m, n \in \mathbb{N}$.

Definition (8), (15):

The Laplace transform, of the following derivative of order $\alpha > 0$ is

$\mathcal{L}\{{}_0^R D_t^\alpha h(t); \lambda\} = \lambda^\alpha F(\lambda) - \sum_{k=0}^{n-1} \lambda^k [{}_0^R D_t^{\alpha-k-1} h(t)]_{t=0}$

$n-1 \le \alpha < n$, where $F = \mathcal{L}(h(t))$.

Lemma (1), (17):
If $R(t)$ is a linear operator and $I^{1-\alpha}R(t)x \in C^1([0,T]), T>0$ then,

$${}^{R}D^{\alpha}\int_0^t R(t-s)x\,ds = \int_0^t {}^{R}D^{\alpha}R(t-s)x\,ds + \lim_{t\to 0^+} I^{1-\alpha}R(t)x,$$

$x \in X, \quad t\in[0,T]$, for $0<\alpha<1$.

Lemma (2), (17):
If g is a continuous function and $I^{1-\alpha}g(t)\in C^1([0,T]),\ T>0$ and $R(t)$ is continuous, then

$${}^{R}D^{\alpha}\int_0^t R(t-s)g(s)ds = \int_0^t R(t-s)\,{}^{R}D^{\alpha}g(s)ds,$$

$t\in[0,T]$, for $0<\alpha<1$.

Now, assume the following conditions:

(A_1) $P=(I-\beta A^{\alpha})$ and $\tilde{A}^{\alpha}=\left(\omega_1+\omega_2 M\left[\left\|A^{\alpha/2}z\right\|^2\right]\right)A^{\alpha}-A^{2\alpha}$, are closed linear operators, $D(P)\subset D(\tilde{A}^{\alpha})$, P is bijective and $P^{-1}: X\to D(P)$ is continuous.

(A_2) $S_{\alpha_1+\alpha_2}(t),\ t\geq 0$ is a strongly continuous semigroup generated by $P^{-1}\tilde{A}^{\alpha}$ on $C_{\nu}^{RL}([0,T],X)$.

(A_3) $S_{\alpha_1+\alpha_2}(t),\ t>0$ is compact operator.

(A_4) The function $f: J\times C_{\nu}^{RL}(J,X)\times C_{\nu}^{RL}(J,X)\to C_{\nu}^{RL}(J,X)$ satisfies

(i) for every $\left(z(t,x),\ {}^{R}D_t^{\gamma}z(t,x)\right)\in C_{\nu}^{RL}(J,X)\times C_{\nu}^{RL}(J,X)$, the function $f(.,z(t,x),\ {}^{R}D_t^{\gamma}z(t,x)): J\to C_{\nu}^{RL}(J,X)$ is strongly measurable.

(ii) $f(t,.,.): C_{\nu}^{RL}(J,X)\times C_{\nu}^{RL}(J,X)\to C_{\nu}^{RL}(J,X)$ is continuous for all $t\in[0,T]=J$.

(iii) $\left\|f(t,z,\ {}^{R}D_t^{\gamma}z)\right\|\leq K_f(t)\Omega_f\left(\|z\|+\left\|{}^{R}D_t^{\gamma}z\right\|\right)$, for $(t,z,\ {}^{R}D_t^{\gamma}z)\in J\times C_{\nu}^{RL}([0,T],X)\times C_{\nu}^{RL}([0,T],X)$ where $K_f(t)$ nonnegative continuous function and Ω_f nondecreasing continuous positive function.

(A_5) The function $g:[0,T]\times X\times X\to X$ satisfies

(i) $g(t,.,.): X\times X\to X$ is continuous for all $t\in[0,T]$.

(ii) $g(.,z,I_t^{\gamma_2}z): X\times X\to X$ is measurable for all $z, I_t^{\gamma_2}z\in C([0,T],X)$.

(iii) $\left\|g(t,z,I_t^{\gamma_2}z)\right\|\leq K_g(t)\Omega_g\left(\|z\|+\left\|I_t^{\gamma_2}z\right\|\right)$ for $(t,z,I_t^{\gamma_2}z)\in J\times X\times X$.

where $K_g(t)$ non-negative continuous function and Ω_g nondecreasing continuous positive function.

(A_6) let

(i) $$\widehat{K}_1=\left\{\left(\tilde{\tilde{N}}+\tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\,t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{z}_0\| +\lambda\left(\tilde{\tilde{N}}+\tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\,t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{y}_0\| +\lambda^{\alpha_1}\left(\tilde{\tilde{N}}+\tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\,t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{v}_0\|\right.$$

(ii) $\widehat{K}_3=max\left\{1,\frac{T^{\nu-\gamma_1}}{\Gamma(\nu-\gamma_1+1)}\right\}max\left\{1,\frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}\right\}$ and $\widehat{K}_2=\widehat{K}_1\widehat{K}_3$

(iii) $\hat{L}(s)=\|P^{-1}\|\int_0^t\left(\tilde{M}_{\alpha_1+\alpha_2}e^{w(t-s)^{\alpha_1+\alpha_2}}\|(t-s)^{\alpha_1+\alpha_2-1}\|+\tilde{\tilde{N}}\right)K_f(s)\Omega_f\,K_g(s)\Omega_g$

Existence of solution:

Now, to prove the existence of a mild solution for problem (1.3) in the space $C_{\nu}^{RL}([0,T],X)=\left\{z\in C([0,T]):\ {}_{R}D_t^{\nu}z\in C([0,T]):\ {}^{R}D_t^{\alpha_1}\,{}^{R}D_t^{\alpha_2}z\in C([0,T])\right\}$ with norm $\|z\|^*=\|z\|_C+\left\|{}^{R}D_t^{\nu}z\right\|_C$, ($\|.\|_C$ is sup norm of $C([0,T],X)$, $\nu=max\{\alpha,\alpha_1+\alpha_2,\gamma_1\}$).

From (A_1) the equation (1.3) can be rewrite as follows:

$$\begin{cases} {}^{R}D_t^{\alpha_1}\,{}^{R}D_t^{\alpha_2}z(t,x) = P^{-1}\tilde{A}^{\alpha}z(t,x) + P^{-1}f\left[t,z(t,x),{}^{R}D_t^{\gamma_1}z(t,x)\right] g\left[t,z(t,x),I_t^{\gamma_2}z(t,x)\right] \\ {}^{R}D_t^{\alpha_1+\alpha_2-1}\,z(t,x)\big|_{t=0}=\tilde{z}_0 \\ {}^{R}D_t^{\alpha_1+\alpha_2-2}\,z(t,x)\big|_{t=0}=\tilde{y}_0 \\ {}^{R}D_t^{\alpha_2-1}\,z(t,x)\big|_{t=0}=\tilde{v}_0 \end{cases} \qquad (3.4)$$

Remark (3), implies that

$${}^{R}D_t^{\alpha_1+\alpha_2}z(t,x)-\sum_{j=1}^{m}\left[{}^{R}D_t^{\alpha_2-j}\,z(t,x)\right]\Big|_{t=0}\frac{t^{-\alpha_1-j}}{\Gamma(1-\alpha_1-j)} = P^{-1}\tilde{A}^{\alpha}z(t,x)$$

$$+P^{-1}f\left[t,z(t,x),\ {}^{R}D_t^{\gamma_1}z(t,x)\right]g\left[t,z(t,x),I_t^{\gamma_2}z(t,x)\right]$$

$$\mathcal{L}\left[{}^{R}D_t^{\alpha_1+\alpha_2}z(t,x)\right]=\sum_{j=1}^{m}\left[{}^{R}D_t^{\alpha_2-j}\,z(t,x)\right]\Big|_{t=0}\mathcal{L}\left[\frac{t^{-\alpha_1-j}}{\Gamma(1-\alpha_1-j)}\right]+P^{-1}\tilde{A}^{\alpha}\mathcal{L}\big(z(t,x)\big)$$
$$+\mathcal{L}\left[P^{-1}f\left[t,z(t,x),\ {}^{R}D_t^{\gamma_1}z(t,x)\right]g\left[t,z(t,x),I_t^{\gamma_2}z(t,x)\right]\right]$$

$$\lambda^{\alpha_1+\alpha_2}\mathcal{L}\big(z(t,x)\big)(\lambda)-\sum_{k=0}^{n-1}\lambda^k\left[{}^{R}D_t^{\alpha_1+\alpha_2-k-1}\,z(t,x)\right]\Big|_{t=0}(\lambda) = \sum_{j=1}^{m}\left[{}^{R}D_t^{\alpha_2-j}\,z(t,x)\right]\Big|_{t=0}\mathcal{L}\left[\frac{t^{-\alpha_1-j}}{\Gamma(1-\alpha_1-j)}\right] +P^{-1}\tilde{A}^{\alpha}\mathcal{L}\big(z(t,x)\big)(\lambda) +\mathcal{L}\left[P^{-1}f\left[t,z,\ {}^{R}D_t^{\gamma_1}z\right]g\left[t,z,I_t^{\gamma_2}z\right]\right](\lambda)$$

where $1<\alpha_1+\alpha_2\leq 2$, thus

$$\lambda^{\alpha_1+\alpha_2}\mathcal{L}\big(z(t,x)\big)(\lambda)-\left[{}^{R}D_t^{\alpha_1+\alpha_2-1}\,z(t,x)\right]\Big|_{t=0}(\lambda) -\lambda\left[{}^{R}D_t^{\alpha_1+\alpha_2-2}\,z(t,x)\right]\Big|_{t=0}(\lambda) = \left[{}^{R}D_t^{\alpha_2-1}\,z(t,x)\right]\Big|_{t=0}\lambda^{\alpha_1}+P^{-1}\tilde{A}^{\alpha}\mathcal{L}\big(z(t,x)\big)(\lambda) +\mathcal{L}\left[P^{-1}f\left[t,z,\ {}^{R}D_t^{\gamma_1}z\right]g\left[t,z,I_t^{\gamma_2}z\right]\right](\lambda)$$

$$\left(\lambda^{\alpha_1+\alpha_2}I-P^{-1}\tilde{A}^{\alpha}\right)\mathcal{L}\big(z(t,x)\big)(\lambda)=\tilde{z}_0+\lambda\tilde{y}_0 +\lambda^{\alpha_1}\tilde{v}_0+\mathcal{L}\left[P^{-1}f\left[t,z,\ {}^{R}D_t^{\gamma_1}z\right]g\left[t,z,I_t^{\gamma_2}z\right]\right](\lambda) \qquad (3.5)$$

Condition (A_2) for $\lambda^{\alpha_1+\alpha_2} \in \rho(P^{-1}\tilde{A}^{\alpha})$ implies that $(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}$

Multiply (3.5) by $(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}$ of both sides

$$\mathcal{L}(z(t,x))(\lambda) = (\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0 + \lambda(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{y}_0 + \lambda^{\alpha_1}(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{v}_0 + (\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1} (\mathcal{L}[P^{-1}f[t,z(t,x), {}^{R}D_t^{\gamma_1}z(t,x)] \; g[t,z(t,x), I_t^{\gamma_2}z(t,x)]]) \quad (3.6)$$

Lemma (3):

If $S_{\alpha_1+\alpha_2}(t), t \geq 0$ strongly continuous semigroup in X generated by $P^{-1}\tilde{A}^{\alpha}$ and $\lambda^{\alpha_1+\alpha_2} \in \rho(P^{-1}\tilde{A}^{\alpha})$. Then

i) for every function $z(t,x) \in$ L([0,T] ; X),

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\mathcal{L}(z(t,x))(\lambda) = \int_0^\infty e^{-\lambda t}\left[\int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1}z(s,x)ds\right]dt \quad (3.7)$$

ii) for every $\tilde{z}_0 \in D(P^{-1}\tilde{A}^{\alpha})$,

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0 = \int_0^\infty e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)t^{\alpha_1+\alpha_2-1}\tilde{z}_0 dt \quad (3.8)$$

iii) for every $\tilde{y}_0 \in D(P^{-1}\tilde{A}^{\alpha})$, $\lambda^{\alpha_1+\alpha_2} \in \rho(P^{-1}\tilde{A}^{\alpha})$ and λ complex number,

$$\lambda(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{y}_0 = \lambda\int_0^\infty e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\,t^{\alpha_1+\alpha_2-1}\tilde{y}_0 dt \quad (3.9)$$

iv) for every $\tilde{y}_0 \in D(P^{-1}\tilde{A}^{\alpha})$, $\lambda^{\alpha_1+\alpha_2} \in \rho(P^{-1}\tilde{A}^{\alpha})$

$$\lambda^{\alpha_1}(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{v}_0 = \lambda^{\alpha_1}\int_0^\infty e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\,t^{\alpha_1+\alpha_2-1}\tilde{v}_0 dt$$

where, $1 < \alpha \leq 2$, $1 < \alpha_1+\alpha_2 \leq 2$ and

$$\tilde{S}_{\alpha_1+\alpha_2}(t)x = \int_0^\infty(\alpha_1+\alpha_2)rM_\alpha(r)\,S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}r)x\,dr$$

where M_α is defined in Definition (3).

Proof:

i) Remark (2), for $z \in D(P^{-1}\tilde{A}^{\alpha})$, implies that

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\mathcal{L}(z(t,x))(\lambda) = \int_0^\infty e^{-\lambda^{\alpha_1+\alpha_2}u}\,S_{\alpha_1+\alpha_2}(u)(\mathcal{L}z(t,x))(\lambda)du$$

Applying Laplace transform of righty side , implies that

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}(\mathcal{L}z(t,x))(\lambda) = \int_0^\infty\int_0^\infty e^{-\lambda^{\alpha_1+\alpha_2}u}\,S_{\alpha_1+\alpha_2}(u)\,e^{-s\lambda}z(t,x)dsdu$$

assume that $u = x^{\alpha_1+\alpha_2}$ then

$du = (\alpha_1+\alpha_2)x^{\alpha_1+\alpha_2-1}dx$,

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}(\mathcal{L}z(t,x))(\lambda) = \int_0^\infty\int_0^\infty e^{-(\lambda x)^{\alpha_1+\alpha_2}}S_{\alpha_1+\alpha_2}(x^{\alpha_1+\alpha_2})(\alpha_1+\alpha_2)x^{\alpha_1+\alpha_2-1} e^{-s\lambda}z(t,x)dsdx$$

Remark (1),

$$e^{-(\lambda x)^{\alpha_1+\alpha_2}} = \int_0^\infty \frac{\alpha_1+\alpha_2}{r^{\alpha_1+\alpha_2+1}}\,M_\alpha(r^{-(\alpha_1+\alpha_2)})e^{-\lambda rx}dr$$

implies that

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}(\mathcal{L}z(t,x))(\lambda) = \int_0^\infty\int_0^\infty\int_0^\infty e^{-\lambda rx}\frac{\alpha_1+\alpha_2}{r^{\alpha_1+\alpha_2+1}}M_\alpha(r^{-\alpha_1+\alpha_2}) S_{\alpha_1+\alpha_2}(x^{\alpha_1+\alpha_2})(\alpha_1+\alpha_2)x^{\alpha_1+\alpha_2-1}e^{-\lambda s}z(t,x)drdsdx$$

now assume that $rx = t$, hence $rdx = dt$, so

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}(\mathcal{L}z(t,x))(\lambda) = \int_0^\infty\int_0^\infty\int_0^\infty e^{-\lambda t}\frac{(\alpha_1+\alpha_2)^2}{r^{\alpha_1+\alpha_2+1}}M_\alpha(r^{-(\alpha_1+\alpha_2)}) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}r^{-(\alpha_1+\alpha_2)})t^{\alpha_1+\alpha_2-1}r^{-(\alpha_1+\alpha_2)} e^{-\lambda s}z(s)dr\,ds\,dt$$

Assume that $y = r^{-(\alpha_1+\alpha_2)}$ hence $r = y^{\frac{-1}{\alpha_1+\alpha_2}}$ and $dr = \frac{-1}{\alpha_1+\alpha_2}\,y^{\frac{-1}{\alpha_1+\alpha_2}-1}dy$, then

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}(\mathcal{L}z(t,x))(\lambda) = \int_0^\infty\int_0^\infty\int_0^\infty e^{-\lambda(t+s)}\frac{(\alpha_1+\alpha_2)^2}{y^{\frac{-1}{\alpha_1+\alpha_2}-1}}M_\alpha(y) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}y)t^{\alpha_1+\alpha_2-1} yz(s)\frac{-1}{\alpha_1+\alpha_2}y^{\frac{-1}{\alpha_1+\alpha_2}-1}dy\,ds\,dt$$

$$= \int_0^\infty e^{-\lambda t}[\int_0^t\int_0^\infty(\alpha_1+\alpha_2)yM_\alpha(y)(t-s)^{\alpha_1+\alpha_2-1} S_{\alpha_1+\alpha_2}((t-s)y)z(s)dy\,ds\,]dt$$

$$= \int_0^\infty e^{-\lambda t}\left[\int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1}z(s)ds\right]dt$$

ii) Remark (2), for any $\tilde{z}_0 \in D(P^{-1}\tilde{A}^{\alpha})$, implies that

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0 = \int_0^\infty e^{-\lambda^{\alpha_1+\alpha_2}u}S_{\alpha_1+\alpha_2}(u)\tilde{z}_0 du$$

assume that $u = x^{\alpha_1+\alpha_2}$ hence

$du = \alpha_1+\alpha_2 x^{\alpha_1+\alpha_2-1}dx$. Then

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0 = \int_0^\infty e^{-(\lambda x)^{\alpha_1+\alpha_2}}S_{\alpha_1+\alpha_2}(x^{\alpha_1+\alpha_2}) (\alpha_1+\alpha_2)x^{\alpha_1+\alpha_2-1}\tilde{z}_0 dx$$

From remark (1), yield

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0 = \int_0^\infty\int_0^\infty\frac{\alpha_1+\alpha_2}{r^{\alpha_1+\alpha_2+1}}e^{-\lambda rx}M_\alpha(r^{-(\alpha_1+\alpha_2)}) S_{\alpha_1+\alpha_2}(x^{\alpha_1+\alpha_2})(\alpha_1+\alpha_2)x^{\alpha_1+\alpha_2-1}\tilde{z}_0 drdx$$

Again, assume that $rx = t$ hence $rdx = dt$, therefore

$$(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha})^{-1}\tilde{z}_0 = \int_0^\infty\int_0^\infty e^{-\lambda t}\frac{(\alpha_1+\alpha_2)^2}{r^{\alpha_1+\alpha_2+1}}M_\alpha(r^{-(\alpha_1+\alpha_2)}) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}r^{-(\alpha_1+\alpha_2)}) t^{\alpha_1+\alpha_2-1}r^{-(\alpha_1+\alpha_2)}\,\tilde{z}_0 drdt$$

Now, assume that $y = r^{-(\alpha_1+\alpha_2)}$ implies $r = y^{\frac{-1}{\alpha_1+\alpha_2}}$ and $dr = \frac{-1}{\alpha_1+\alpha_2}\,y^{\frac{-1}{\alpha_1+\alpha_2}-1}dy$. Then

$\left(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha}\right)^{-1}\tilde{z}_0$
$= \int_0^{\infty}\int_0^{\infty} e^{-\lambda t}\,(\alpha_1+\alpha_2) y M_{\alpha}(y)$
$S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}y)t^{\alpha_1+\alpha_2-1}\tilde{z}_0 dy dt$
$\left(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha}\right)^{-1}\tilde{z}_0$
$= \int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0 dt$

iii) From remark (2), for every $\tilde{y}_0 \in D(P^{-1}A^{\alpha})$, the prove (iii) is a same way of the prove (ii) ,yield
$\left(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha}\right)^{-1}\tilde{y}_0$
$= \int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 dt$
multiple both said by complex number λ ,then
$\lambda\left(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha}\right)^{-1}\tilde{y}_0$
$= \lambda\int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 dt$

iv) From remark (2), for every $\tilde{v}_0 \in D(P^{-1}A^{\alpha})$, the proving of (iv) is a same way of the prove (ii), that is
$\left(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha}\right)^{-1}\tilde{v}_0$
$= \int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 dt$
multiple both said by complex number λ^{α_1} ,then
$\lambda^{\alpha_1}\left(\lambda^{\alpha_1+\alpha_2}I - P^{-1}\tilde{A}^{\alpha}\right)^{-1}\tilde{v}_0$
$= \lambda^{\alpha_1}\int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 dt.$

Now, by lemma (3) the equation (3.6) take the form:

$$\mathcal{L}\big(z(t,x)\big)(\lambda) = \int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0 dt + \lambda\int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 dt + \lambda^{\alpha_1}\int_0^{\infty} e^{-\lambda t}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 dt + \int_0^{\infty} e^{-\lambda t}\Big[\int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1} P^{-1}f\big[t,z(t,x),\, {}^{R}D_t^{\gamma_1}z(t,x)\big]g\big[t,z(t,x),I_t^{\gamma_2}z(t,x)\big]ds\Big]dt$$

applying the Laplace inverse of the above equation, yield

$$z(t,x) = \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0 + \lambda\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 + \lambda^{\alpha_1}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 + \int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1}\Big[P^{-1}f\big[s,z(s,x),\, {}^{R}D_s^{\gamma_1}z(s,x)\big]g\big[s,z(s,x),I_s^{\gamma_2}z(s,x)\big]\Big]ds \quad (3.10)$$

Definition (9)

A function $z(t,x)\colon [0,T]\times X \to X$ is called a mild solution of (3.4). If $z(t,x)$ and the fractional derivative ${}^{R}D_t^{\gamma_1}z(t,x)$ exists for $0<\gamma_1\le 1$ and both are continuous on $[0,T]$, and satisfies the following equation:

$$z(t,x) = \tilde{S}_{\alpha_1+\alpha_2}(t)t^{\alpha_1+\alpha_2-1}\tilde{z}_0 + \lambda\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 + \lambda^{\alpha_1}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 + \int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1} P^{-1}f\big[s,z(s,x),\, {}^{R}D_S^{\gamma_1}z(s,x)\big] g\big[s,z(s,x),I_s^{\gamma_2}z(s,x)\big]ds, \quad t\in[0,T].$$

Lemma (4):
Let $S_{\alpha_1+\alpha_2}(t), t \ge 0$ be a strongly continuous semigroup in X, generated by $P^{-1}\tilde{A}^{\alpha_1+\alpha_2}$. Then the family of operators $\{\tilde{S}_{\alpha_1+\alpha_2}(t), t \ge 0\}$ and $1<\alpha\le 2$ satisfy the following.

i) For any $t \ge 0$, $\tilde{S}_{\alpha_1+\alpha_2}(t)$ is bounded linear operator (for any $x\in X$ there exists $\widetilde{M}_{\alpha_1+\alpha_2} > 1$, $w\ge 0$ then $\|\tilde{S}_{\alpha_1+\alpha_2}(t)\| \le \widetilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}$.
ii) The family $\{\tilde{S}_{\alpha_1+\alpha_2}(t), t\ge 0\}$ is strongly continuous, which means that for every $y\in X$ and $0\le t_1<t_2\le T$, then
$\|\tilde{S}_{\alpha_1+\alpha_2}(t_2)x - \tilde{S}_{\alpha_1+\alpha_2}(t_1)y\|_Z \to 0$ when $t_2\to t_1$.
iii) If $S_{\alpha_1+\alpha_2}(t)$ is a compact, then the operator $\tilde{S}_{\alpha_1+\alpha_2}(t)$ is a compact in X, $t>0$.

proof:

i. For any fixed ≥ 0 , since $S_{\alpha_1+\alpha_2}(t)$ is a linear operator , then $\tilde{S}_{\alpha_1+\alpha_2}(t)$ is also linear operator. For any $y\in X$, and from lemma (3),thus
$\|\tilde{S}_{\alpha_1+\alpha_2}(t)y\|_Z$
$= \left\|\int_0^{\infty}\alpha_1+\alpha_2 r M_{\alpha_1+\alpha_2}(r)\, S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}r)y\, dr\right\|_Z$
From properties of strongly continuous semigroup $S_{\alpha_1+\alpha_2}(t)$, there is $\widetilde{M}>1$ and $w\ge 0$, that is
$\|\tilde{S}_{\alpha_1+\alpha_2}(t)y\|_Z$
$\le \alpha_1+\alpha_2\widetilde{M}\int_0^{\infty} r M_{\alpha_1+\alpha_2}(r)\, e^{wt^{\alpha_1+\alpha_2}r}\|y\|_X dr$
$\le \alpha_1+\alpha_2\widetilde{M}\int_0^{\infty} r M_{\alpha_1+\alpha_2}(r)\sum_{n=0}^{\infty}\frac{(wt^{\alpha_1+\alpha_2}r)^n}{n!}\|y\|_X\, dr$
$\le (\alpha_1+\alpha_2)\widetilde{M}\sum_{n=0}^{\infty}\frac{(wt^{\alpha_1+\alpha_2})^n}{n!}\int_0^{\infty} r^{n+1}M_{\alpha_1+\alpha_2}(r)\|y\|_X dr$
$\le (\alpha_1+\alpha_2)\widetilde{M}\sum_0^{\infty}\frac{(wt^{\alpha_1+\alpha_2})^n}{n!}\frac{\Gamma(n+2)}{\Gamma((\alpha_1+\alpha_2)(n+1)+1}\|y\|_X$
$\le (\alpha_1+\alpha_2)\widetilde{M}\sum_0^{\infty}\frac{(wt^{\alpha_1+\alpha_2})^n}{n!}\frac{(n+1)\Gamma(n+1)}{(\alpha_1+\alpha_2)(n+1)\Gamma((\alpha_1+\alpha_2)(n+1))}\|y\|_X \le \widetilde{M}\,E_{\alpha,\alpha}(wt^{\alpha})\|y\|_X$
where $E_{\alpha,\alpha}$ is a Mittag-Leffler function.
From properties of Mittag-Leffler function,
$\|\tilde{S}_{\alpha_1+\alpha_2}(t)y\|_Z \le \widetilde{M}\widehat{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|y\|_Z$,
where $1 < \widehat{M}_{\alpha_1+\alpha_2} = \sup_{n\in N}\frac{n!}{\Gamma((\alpha_1+\alpha_2)n+(\alpha_1+\alpha_2))}$
Assume that $\widetilde{M}_{\alpha_1+\alpha_2} = \widetilde{M}\widehat{M}_{\alpha_1+\alpha_2} > 1$, then
$\|\tilde{S}_{\alpha_1+\alpha_2}(t)y\|_X \le \widetilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|y\|_X$.

ii. For any $y\in X$ and $0\le t_1<t_2\le T$, then
$\|\tilde{S}_{\alpha_1+\alpha_2}(t_2)x - \tilde{S}_{\alpha_1+\alpha_2}(t_1)y\|_X$

$= \left\|\int_0^{\infty}(\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r)[S_{\alpha_1+\alpha_2}((t_2)^{\alpha_1+\alpha_2}r) - S_{\alpha_1+\alpha_2}((t_1)^{\alpha_1+\alpha_2}r)]y dr\right\|_X$

$\le \int_0^{\infty}(\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r)\|S_{\alpha_1+\alpha_2}((t_2)^{\alpha_1+\alpha_2}r) - S_{\alpha_1+\alpha_2}((t_1)^{\alpha_1+\alpha_2}r)\|_X\|y\|_X dr$

According to the strongly continuity of $S_{\alpha_1+\alpha_2}(t)$, $t\ge 0$, thus $\|S_{\alpha_1+\alpha_2}(t_2)z - S_{\alpha_1+\alpha_2}(t_1)y\|_X$ tends to zero as $t_2 \to t_1$, which means that $\{\tilde{S}_{\alpha_1+\alpha_2}(t), t \geq 0\}$ is strongly continuous.

iii. To prove that $\tilde{S}_{\alpha_1+\alpha_2}(t)$ is compact in X, $t \geq 0$. For each positive constant L, the set $Z_L = \{x \in X : \|x\| \leq L\}$ is clearly a bounded subset in X . To prove that for any positive constant L and $t \geq 0$, the set

$W(t) = \{\tilde{S}_{\alpha_1+\alpha_2}(t)x, x \in Z_L\}$
$= \{\int_0^\infty (\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2} r)x\, dr, x \in Z_L\}$

is relatively compact in X. Let $t \geq 0$, for all $\delta > 0$, the following subset in X ;

$W_\delta(t) =$
$\{\int_\delta^\infty (\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2} r)x\, dr, x \in Z_L\}$,

Then for $x \in Z_L$, implies that

$$\int_\delta^\infty (\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2} r)x\, dr = S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}\delta) \int_\delta^\infty (\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2} r - t^{\alpha_1+\alpha_2}\delta)x\, dr$$

Since $S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2}\delta)$, $t^{\alpha_1+\alpha_2}\delta > 0$ is a compact, for any $t > 0$ the $W_\delta(t)$ is relatively compact in X. Moreover, for every $x \in Z_L$, thus

$$\left\| \int_0^\infty (\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2} r)x\, dr - \int_\delta^\infty (\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2} r)x\, dr \right\|$$
$$= \left\| \int_0^\delta (\alpha_1+\alpha_2) r M_{\alpha_1+\alpha_2}(r) S_{\alpha_1+\alpha_2}(t^{\alpha_1+\alpha_2} r)x\, dr \right\| \leq (\alpha_1+\alpha_2) M \int_0^\delta e^{t^{\alpha_1+\alpha_2} r} r M_{\alpha_1+\alpha_2}(r)\, dr$$

There are arbitrary relatively compact sets close to the set $W(t)$, $t > 0$. Hence the set $W(t)$, $t > 0$ is also relatively compact in X.

Theorem (1):

Consider (A_1)-(A_4) holds and

$$\widehat{K}_3 \int_0^{\hat{t}} \hat{L}(s)\, ds < \int_{\widehat{K}_2}^\infty \frac{ds}{\Omega_f(s)\Omega_g(s)} \quad (3.11)$$

where $\widehat{K}_1, \widehat{K}_2, \widehat{K}_3$ and $\hat{L}(s)$ defined in assumption (A_6). Then the problem (3.4), has a mild solution $z \in C_\nu^{RL}([0,T], X)$.

proof:

From our assumptions and $z \in C_\nu^{RL}([0,T], X)$ the maps

$$\Phi(z,y)(t) = \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0 + \lambda \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 + \lambda^{\alpha_1}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 + \int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1} \left[P^{-1} f[s, z(s,x), I_s^{\nu-\gamma_1} y(s,x)]\, g[s, z(s,x), I_s^{\gamma_2} z(s,x)]\right] ds \quad (3.12)$$

for $t \in [0,T]$, and

$$\Psi(z,y)(t) = {}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0 + \lambda\, {}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 + \lambda^{\alpha_1}\, {}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 + {}^R D^\nu \int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1} \left[P^{-1} f[s, z(s,x), I_s^{\nu-\gamma_1} y(s,x)]\, g[s, z(s,x), I_s^{\gamma_2} z(s,x)]\right] ds \quad (3.13)$$

by using Schaefer's theorem, this means that firstly, that the set

$$(z_{\alpha_1+\alpha_2}, y_{\alpha_1+\alpha_2}) = \tau\left(\Phi(z_{\alpha_1+\alpha_2}, y_{\alpha_1+\alpha_2}), \Psi(z_{\alpha_1+\alpha_2}, y_{\alpha_1+\alpha_2})\right), \quad 0 < \tau < 1$$

is bounded. Secondly, to show that the operator $F: C_\nu^{RL}([0,T], X) \to C_\nu^{RL}([0,T], X)$ is completely continuous. Now, from (3.12) that

$$\|z_{\alpha_1+\alpha_2}(t,x)\| \leq \widetilde{M}_{\alpha_1+\alpha_2} e^{wt^{\alpha_1+\alpha_2}} \|t^{\alpha_1+\alpha_2-1}\| \|\tilde{z}_0\| + \lambda \widetilde{M}_{\alpha_1+\alpha_2} e^{wt^{\alpha_1+\alpha_2}} \|t^{\alpha_1+\alpha_2-1}\| \|\tilde{y}_0\| + \|P^{-1}\| \int_0^t \widetilde{M}_{\alpha_1+\alpha_2} e^{w(t-s)^{\alpha_1+\alpha_2}} \|(t-s)^{\alpha_1+\alpha_2-1}\| \|f[s, z(s,x), I_s^{\nu-\gamma_1} y(s,x)]\| \|g[s, z(s,x), I_s^{\gamma_2} z(s,x)]\|\, ds$$
$$\leq \widetilde{M}_{\alpha_1+\alpha_2} e^{wt^{\alpha_1+\alpha_2}} \|t^{\alpha_1+\alpha_2-1}\| \|\tilde{z}_0\| + \lambda \widetilde{M}_{\alpha_1+\alpha_2} e^{wt^{\alpha_1+\alpha_2}} \|t^{\alpha_1+\alpha_2-1}\| \|\tilde{y}_0\| + \|P^{-1}\| \int_0^t \widetilde{M}_{\alpha_1+\alpha_2} e^{w(t-s)^{\alpha_1+\alpha_2}} \|(t-s)^{\alpha_1+\alpha_2-1}\|$$
$$\times K_f(s)\Omega_f\left(\|z_{\alpha_1+\alpha_2}(s)\| + \frac{s^{\nu-\gamma_1}}{\Gamma(\nu-\gamma_1+1)} sup_{0\leq\tau\leq s} \|y_{\alpha_1+\alpha_2}(\tau)\|\right)$$
$$\times K_g(s)\Omega_g\left(\|z_{\alpha_1+\alpha_2}(s)\| + \frac{s^{\gamma_2}}{\Gamma(\gamma_2+1)} sup_{0\leq\tau\leq s} \|y_{\alpha_1+\alpha_2}(\tau)\|\right) ds$$

and from (3.13)

$$\|y_{\alpha_1+\alpha_2}(t,x)\| \leq \|{}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0\| + \|\lambda\, {}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0\| + \|\lambda^{\alpha_1}\, {}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0\| + \left\| {}^R D^\nu \int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1} \times \left[P^{-1} f[s, z(s,x), I_s^{\nu-\gamma_1} y(s,x)]\, g[s, z(s,x), I_s^{\gamma_2} z(s,x)]\right] ds \right\|$$

Let $\widetilde{\widetilde{N}} = \max\|{}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\|$, and from lemma (2), then

$$\|y_{\alpha_1+\alpha_2}(t,x)\| \leq \widetilde{\widetilde{N}}\|\tilde{z}_0\| + \lambda\widetilde{\widetilde{N}}\|\tilde{y}_0\| + \lambda^{\alpha_1}\widetilde{\widetilde{N}}\|\tilde{v}_0\| + \int_0^t \|{}^R D^\nu \tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}\| \|P^{-1}\| \|f[s, z(s,x), I_s^{\nu-\gamma_1} y(s,x)]\| \|g[s, z(s,x), I_s^{\gamma_2} z(s,x)]\| ds$$
$$\leq \widetilde{\widetilde{N}}\|\tilde{z}_0\| + \lambda\widetilde{\widetilde{N}}\|\tilde{y}_0\| + \lambda^{\alpha_1}\widetilde{\widetilde{N}}\|\tilde{v}_0\| + \|P^{-1}\|\widetilde{\widetilde{N}}$$
$$\int_0^t K_f(s)\Omega_f\left(\|z_{\alpha_1+\alpha_2}(s)\| + \frac{s^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} \sup_{0\leq\tau\leq s} \|y_{\alpha_1+\alpha_2}(\tau)\|\right) K_g(s)\Omega_g\left(\|z_{\alpha_1+\alpha_2}(s)\| + \frac{s^{\gamma_2}}{\Gamma(\gamma_2+1)} sup_{0\leq\tau\leq s} \|y_{\alpha_1+\alpha_2}(\tau)\|\right) ds$$

Clearly

$$\|z_{\alpha_1+\alpha_2}(t,x)\| + \|y_{\alpha_1+\alpha_2}(t,x)\| \leq \left(\widetilde{\widetilde{N}} + \widetilde{M}_{\alpha_1+\alpha_2} e^{wt^{\alpha_1+\alpha_2}} \|t^{\alpha_1+\alpha_2-1}\|\right) \|\tilde{z}_0\|$$
$$+ \lambda\left(\widetilde{\widetilde{N}} + \widetilde{M}_{\alpha_1+\alpha_2} e^{wt^{\alpha_1+\alpha_2}} \|t^{\alpha_1+\alpha_2-1}\|\right) \|\tilde{y}_0\|$$
$$+ \lambda^{\alpha_1}\left(\widetilde{\widetilde{N}} + \widetilde{M}_{\alpha_1+\alpha_2} e^{wt^{\alpha_1+\alpha_2}} \|t^{\alpha_1+\alpha_2-1}\|\right) \|\tilde{v}_0\|$$
$$+ \|P^{-1}\| \int_0^t \left(\widetilde{M}_{\alpha_1+\alpha_2} e^{w(t-s)^{\alpha_1+\alpha_2}} \|(t-s)^{\alpha_1+\alpha_2-1}\| + \widetilde{\widetilde{N}}\right)$$

$K_f(s)\Omega_f\left(\|z_\alpha(s)\| + \frac{s^{\nu-\gamma_1}}{\Gamma(\nu-\gamma+1)}\sup_{0\le\tau\le s}\|y_\alpha(\tau)\|\right)$

$K_g(s)\Omega_g\left(\|z_\alpha(s)\| + \frac{s^{\gamma_2}}{\Gamma(\gamma_2+1)} sup_{0\le\tau\le s}\|y_\alpha(\tau)\|\right)\, ds$

put

$$\vartheta_f(t)\vartheta_g(t) = max\left\{1, \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right\}\sup_{0\le\tau\le t}\left(\|z_{\alpha_1+\alpha_2}(\tau)\| + \|y_{\alpha_1+\alpha_2}(\tau)\|\right) max\left\{1, \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}\right\}\sup_{0\le\tau\le\hat{t}}\left(\|z_{\alpha_1+\alpha_2}(\tau)\| + \|y_{\alpha_1+\alpha_2}(\tau)\|\right)$$

Then

$$\vartheta_f(t)\vartheta_g(t) \le max\left\{1, \frac{T^{\nu-\gamma_1}}{\Gamma(\nu-\gamma_1+1)}\right\} max\left\{1, \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}\right\} \times\left\{\left(\tilde{\tilde{N}} + \tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\, t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{z}_0\| + \lambda\left(\tilde{\tilde{N}} + \tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\, t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{y}_0\| + \lambda^{\alpha_1}\left(\tilde{\tilde{N}} + \tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\, t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{v}_0\| + \|P^{-1}\| \int_0^t\left(\tilde{M}_{\alpha_1+\alpha_2}e^{w(t-s)^{\alpha_1+\alpha_2}}\|(t-s)^{\alpha_1+\alpha_2-1}\| + \tilde{\tilde{N}}\right) K_f(s)\Omega_f\left(\vartheta_f(s)\right) K_g(s)\Omega_g\left(\vartheta_g(s)\right)\, ds$$

$$\vartheta_f(t)\vartheta_g(t) \le max\left\{1, \frac{T^{\nu-\gamma_1}}{\Gamma(\nu-\gamma_1+1)}\right\} max\left\{1, \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}\right\} \left\{\hat{K}_1 + \int_0^t \hat{L}(s)\,\Omega_f\left(\vartheta_f(s)\right)\Omega_g\left(\vartheta_g(s)\right)\, ds\right\}$$

$$\le \hat{K}_3\left\{\hat{K}_1 + \int_0^t \hat{L}(s)\left(\Omega_f\left(\vartheta_f(s)\right)\right)\left(\Omega_g\left(\vartheta_g(s)\right)\right)\, ds\right\}$$

$$\le \hat{K}_2 + \hat{K}_3\int_0^t \hat{L}(s)\left(\Omega_f\left(\vartheta_f(s)\right)\right)\left(\Omega_g\left(\vartheta_g(s)\right)\right)\, ds$$

where

$$\hat{K}_1 = \left\{\left(\tilde{\tilde{N}} + \tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\, t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{z}_0\| + \lambda\left(\tilde{\tilde{N}} + \tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\, t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{y}_0\| + \lambda^{\alpha_1}\left(\tilde{\tilde{N}} + \tilde{M}_{\alpha_1+\alpha_2}e^{wt^{\alpha_1+\alpha_2}}\|\, t^{\alpha_1+\alpha_2-1}\|\right)\|\tilde{v}_0\|\right.$$

$\hat{K}_3 = max\left\{1, \frac{T^{\nu-\gamma_1}}{\Gamma(\nu-\gamma_1+1)}\right\} max\left\{1, \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}\right\}$ and $\hat{K}_2 = \hat{K}_1\hat{K}_3$

$$\hat{L}(s) = \|P^{-1}\|\int_0^t\left(\tilde{M}_{\alpha_1+\alpha_2}e^{w(t-s)^{\alpha_1+\alpha_2}}\|(t-s)^{\alpha_1+\alpha_2-1}\| + \tilde{\tilde{N}}\right) K_f(s)\Omega_f\, K_g(s)\Omega_g$$

Let $\tilde{\vartheta}_{f,g}(t)$

$$= \hat{K}_2 + \hat{K}_3\int_0^t \hat{L}(s)\left(\Omega_f\left(\vartheta_f(s)\right)\right)\left(\Omega_g\left(\vartheta_g(s)\right)\right)\, ds$$

then

$$\vartheta_f(t)\vartheta_g(t) \le \tilde{\vartheta}_{f,g}(t) \text{ and } \tilde{\vartheta}_{f,g}(0) = \hat{K}_2$$

$$\tilde{\vartheta}'_{f,g}(t) \le \hat{K}_3\hat{L}(t)\left(\Omega_f\left(\vartheta_f(t)\right)\right)\left(\Omega_g\left(\vartheta_g(t)\right)\right)$$

Then,

$$\int_{\hat{K}_2}^{\tilde{\vartheta}_{f,g}(t)}\frac{ds}{\Omega_f(s)\Omega_g(s)} \le \hat{K}_3\int_0^t \hat{L}(s)\, ds$$

From (3.13), that is

$$\int_{\hat{K}_2}^{\tilde{\vartheta}_{f,g}(t)}\frac{ds}{\Omega_f(s)\Omega_g(s)} \le \hat{K}_3\int_0^t \hat{L}(s)\, ds < \int_{\hat{K}_2}^{\infty}\frac{ds}{\Omega_f(s)\Omega_g(s)}$$

This means that $\vartheta_{f,g}(t)$ is bounded, then the set of solutions $\left(z_{\alpha_1+\alpha_2}, y_{\alpha_1+\alpha_2}\right) = \tau\left(\Phi\left(z_{\alpha_1+\alpha_2}, y_{\alpha_1+\alpha_2}\right), \Psi\left(z_{\alpha_1+\alpha_2}, y_{\alpha_1+\alpha_2}\right)\right)$, is bounded. (3.14)

Now, to prove that the operator $F: C_\nu^{RL}([0,T], X) \to C_\nu^{RL}([0,T], X)$ is completely continuous, where

$$(Fz)(t) = \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0 + \lambda\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 + \lambda^{\alpha_1}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0 + \int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1}\left[P^{-1}f\left[s, z, {}^{R}D_s^{\gamma_1}z\right]g\left[s, z, I_s^{\gamma_2}z\right]\right] ds$$

Let $H_l = \{z \in C_\nu^{RL}([0,T], X) :\ \|z\|^* \le l, l \ge 1\}$.

The first thing to do is that F maps H_l into an equicontinuous family. Let $z \in H_l$ and $t_1, t_2 \in [0,T]$ then if $0 < t_1 < t_2 \le T$.

$$\|(Fz)(t_1) - (Fz)(t_2)\| \le \left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - \tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\right\|\|\tilde{z}_0\| + \lambda\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - \tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\right\|\|\tilde{y}_0\| + \int_0^{t_1}\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1-s)(t_1-s)^{\alpha_1+\alpha_2-1} - \tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1} P^{-1}f\left[s, z, {}^{R}D_s^{\gamma_1}z\right]g\left[s, z, I_s^{\gamma_2}z\right]\right\| ds + \int_{t_1}^{t_2}\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1} P^{-1}f\left[s, z, {}^{R}D_s^{\gamma_1}z\right]g\left[s, z, I_s^{\gamma_2}z\right]\right\| ds$$

$$\le \left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1) - \tilde{S}_{\alpha_1+\alpha_2}(t_2)\right\|\|\tilde{z}_0\| + \lambda\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1) - \tilde{S}_{\alpha_1+\alpha_2}(t_2)\right\|\|\tilde{y}_0\| + \int_0^{t_1}\left\|\left[\tilde{S}_{\alpha_1+\alpha_2}(t_1-s) - \tilde{S}_{\alpha_1+\alpha_2}(t_2-s)\right] ds\; P^{-1}f\left[s, z, {}^{R}D_s^{\gamma_1}z\right]g\left[s, z, I_s^{\gamma_2}z\right] ds\right.$$

$$\le \left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1) - \tilde{S}_{\alpha_1+\alpha_2}(t_2)\right\|\|\tilde{z}_0\| + \lambda\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1) - \tilde{S}_{\alpha_1+\alpha_2}(t_2)\right\|\|\tilde{y}_0\| + \int_0^{t_1}\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1-s) - \tilde{S}_{\alpha_1+\alpha_2}(t_2-s)\right\|\|P^{-1}\| K_f(s)\Omega_f\left(\|z(s)\| + \left\|I_s^{\nu-\gamma}y(s)\right\|\right) K_g(s)\Omega_g\left(\|z(s)\| + \left\|I_s^{\gamma_2}y(s)\right\|\right) ds + \int_{t_1}^{t_2}\left\|\left[\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)\right]\right\|\,\|P^{-1}\| K_f(s)\Omega_f\left(\|z(s)\| + \left\|I_s^{\nu-\gamma}y(s)\right\|\right) K_g(s)\Omega_g\left(\|z(s)\| + \left\|I_s^{\gamma_2}y(s)\right\|\right) ds$$

$$\le \left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1) - \tilde{S}_{\alpha_1+\alpha_2}(t_2)\right\|\|\tilde{z}_0\| + \lambda\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_1) - \tilde{S}_{\alpha_1+\alpha_2}(t_2)\right\|\|\tilde{y}_0\| + \int_0^{t_1}\left\|\tilde{S}_{\alpha_1+\alpha_2}(t_2-s) - \tilde{S}_{\alpha_1+\alpha_2}(t_2-s)\right\|\|P^{-1}\| K_f(s)\Omega_f\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right) K_g(s)\Omega_g\left(r + \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}r\right) ds + \int_{t_2}^{t_2}\left\|\left[\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)\right]\right\|\,\|P^{-1}\| K_f(s)\Omega_f\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right) K_g(s)\Omega_g\left(r + \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}r\right) ds \quad (3.15)$$

and similarly

$\left\| {}^{R}D_t^{\nu}(Fz)(t_1) - {}^{R}D_t^{\nu}(Fz)(t_2)\right\|$
$\leq \| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{z}_0\|$
$+\lambda\| {}^{R}D_t^{\alpha_1+\alpha_2}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{y}_0\|$
$+\int_0^{t_1}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1-s)(t_1-s)^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]$
$\times\left[P^{-1}f\left[s,z,{}^{R}D_s^{\gamma_1}z(s,x)\right]g\left[s,z,I_s^{\gamma_2}z(s,x)\right]\right]\| \,ds$
$+\int_{t_1}^{t_2}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]$
$\left[P^{-1}f\left[t,z,{}^{R}D_s^{\gamma_1}z(s,x)\right]g\left[s,z,I_s^{\gamma_2}z(s,x)\right]\right]\| \,ds$
$\leq \| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{z}_0\|$
$+\lambda\| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{y}_0\|$
$+\int_0^{t_1}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1-s)(\dot{t}_1-s)^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]$
$\times\left[P^{-1}f\left[s,z,{}^{R}D_s^{\gamma_1}z(s,x)\right]g\left[s,z,I_s^{\gamma_2}z(s,x)\right]\right]\| \,ds$
$+\int_{t_1}^{t_2}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]ds$
$P^{-1}f\left[s,z,I_s^{\nu-\gamma_1}y(s,x)\right]g\left[s,z,I_s^{\gamma_2}z(s,x)\right]\|ds$
$\leq \| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{z}_0\|$
$+\lambda\| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{y}_0\|$
$+\int_0^{t_1}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1-s)(t_1-s)^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]\|$
$\times\|P^{-1}\|K_f(s)\Omega_f\left(\|z(s)\| + \|I_s^{\nu-\gamma}y(s)\|\right)$
$K_g(s)\Omega_g\left(\|z(s)\| + \|I_s^{\gamma_2}y(s)\|\right)ds$
$+\int_{t_1}^{t_2}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]\|$
$\times\|P^{-1}\|K_f(s)\Omega_f\left(\|z(s)\| + \|I_s^{\nu-\gamma}y(s)\|\right)$
$K_g(s)\Omega_g\left(\|z(s)\| + \|I_s^{\gamma_2}y(s)\|\right)ds$
$\leq \| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{z}_0\|$
$+\lambda\| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1)t_1^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2)t_2^{\alpha_1+\alpha_2-1}\|\|\tilde{y}_0\|$
$+\int_0^{t_1}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_1-s)(t_1-s)^{\alpha_1+\alpha_2-1} - {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]\|$
$\times\|P^{-1}\|K_f(s)\Omega_f\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right)$
$K_g(s)\Omega_g\left(r + \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}r\right)ds$
$+\int_{t_1}^{t_2}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t_2-s)(t_2-s)^{\alpha_1+\alpha_2-1}\right]\|$
$\times\|P^{-1}\|K_f(s)\Omega_f\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right)$
$K_g(s)\Omega_g\left(r + \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}r\right)ds$ (3.16)

in equation (3.15) and (3.16) the right-hand side are independent of $z \in H_l$ and tend to zero as $t_1 \to t_2$. Since $\tilde{S}_{\alpha_1+\alpha_2}(t)$, ${}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t)$ are uniformly continuous for $t \in [0,T]$ and the compactness of $S_{\alpha_1+\alpha_2}(t)$, ${}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}$ for $t > 0$ imply the continuity in the uniform operator topology.

Then F maps H_l into an equicontinuous family. Now to show that $\overline{FH_l}$ is compact. Since have been proved that F map H_l into an equicontinuous family. To prove that F maps H_l into a precompact set in $C_\nu^{RL}([0,T],X)$ and apply the Arzela- Ascoli theorem.

Let $\in (0,T]$, $\epsilon \in (0,t)$ for $z \in H_l$, then

$(F_\epsilon z)(t) = \tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{z}_0$
$+\lambda\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{y}_0 + \lambda^{\alpha_1}\tilde{S}_{\alpha_1+\alpha_2}(t)\, t^{\alpha_1+\alpha_2-1}\tilde{v}_0$
$+\int_0^{t-\epsilon}\tilde{S}_{\alpha_1+\alpha_2}(t-s)\,(t-s)^{\alpha_1+\alpha_2-1}$
$\left[P^{-1}f\left[s,z,{}^{R}D_s^{\gamma_1}z(s,x)\right]g\left[s,z,I_s^{\gamma_2}z(s,x)\right]\right]ds$

Since $\tilde{S}_{\alpha_1+\alpha_2}(t)$, ${}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}$ are compact operator, the set $E_\epsilon(t) = \{(F_\epsilon z)(t): z \in H_l\}$ is precompact in $C_\nu^{RL}([0,T],X)$ for $\epsilon \in (0,t)$ and for every $\in H_l$, then

$\|(Fz)(t) - (F_\epsilon z)(t)\|$
$\leq \int_{t-\epsilon}^{t}\|\left[\tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}\right]$
$P^{-1}f\left[s,z,{}^{R}D_s^{\gamma_1}z(s,x)\right]g\left[s,z,I_s^{\gamma_2}z(s,x)\right]\|ds$
$\leq \int_{t-\epsilon}^{t}\|\left[\tilde{S}_{\alpha_1+\alpha_2}(t-s)\right]\|$
$\|P^{-1}\|K_f(s)\Omega_f\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right)$
$K_g(s)\Omega_g\left(r + \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}r\right)ds\,.$ (3.17)

and

$\left\| {}^{R}D_t^{\nu}(Fz)(t) - {}^{R}D_t^{\nu}(F_\epsilon z)(t)\right\|$
$\leq \int_{t-\epsilon}^{t}\|\left[{}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}\right]$
$P^{-1}f\left(s,z(s),{}^{R}D_s^{\gamma_1}z(s)\right)g\left(s,z(s),I_s^{\gamma_2}z(s)\right)\|\,ds$
$\leq \int_{t-\epsilon}^{t}\| {}^{R}D_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}\|$
$\|P^{-1}\|K_f(s)\Omega_f\left(r + \frac{T^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}r\right)$
$K_g(s)\Omega_g\left(r + \frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}r\right)ds.$ (3.18)

Then there are arbitrarily pre-compact sets close to the set $\{(Fz)(t): z \in H_l\}$. Hence the set $\{(Fz)(t): z \in H_l\}$ is precompact in $C_\nu^{RL}([0,T],X)$.

Now, to show that $F: C_\nu^{RL}([0,T],X) \to C_\nu^{RL}([0,T],X)$ is continuous.

Let z_n be a sequence, such that $\{z_n\}_0^\infty \subseteq C_\nu^{RL}([0,T],X)$ with $z_n \to z$ in $C_\nu^{RL}([0,T],X)$. Then there is integer p such that $\|z_n(t,x)\| \leq p$, $\left\| {}^{R}D_t^{\gamma}z_n(t,x)\right\| \leq p$ for all n and $t \in [0,T]$ so $\|z(t,x)\| \leq p$, $\left\| {}^{R}D_t^{\gamma}z(t,x)\right\| \leq p$,

$z(t,x)$, ${}^{R}D_t^{\gamma}z(t,x) \in C_\nu^{RL}([0,T],X)$.

from (A_4),

$$f\left(t,z_n(t,x),{}^{R}D_t^{\gamma_1}z_n(t,x)\right) \longrightarrow f\left(t,z(t,x),{}^{R}D_t^{\gamma_1}z(t,x)\right)$$

and from (A_5), implies that

$$g\left(t,z_n(t,x),I_t^{\gamma_2}z_n(t,x)\right)\to g\left(t,z(t,x),I_t^{\gamma_2}z(t,x)\right)$$

$$f\left(t,z_n(t,x),\ {}^RD_t^{\gamma_1}z_n(t,x)\right)g\left(t,z_n(t,x),I_t^{\gamma_2}z_n(t,x)\right) \to f\left(t,z(t,x),\ {}^RD_t^{\gamma_1}z(t,x)\right)g\left(t,z(t,x),I_t^{\gamma_2}z(t,x)\right)$$

for each $t\in J=[0,T]$ and since

$$\Big\|f\left(t,z_n(t,x),\ {}^RD_t^{\gamma_1}z_n(t,x)\right)g\left(t,z_n(t,x),I_t^{\gamma_2}z_n(t,x)\right) -f\left(t,z(t,x),\ {}^RD_t^{\gamma_1}z(t,x)\right)g\left(t,z(t,x),I_t^{\gamma_2}z(t,x)\right)\Big\|$$
$$\le \Big\|f\left(t,z_n(t,x),\ {}^RD_t^{\gamma_1}z_n(t,x)\right)g\left(t,z_n(t,x),I_t^{\gamma_2}z_n(t,x)\right)\Big\| + \Big\|f\left(t,z(t,x),\ {}^RD_t^{\gamma_1}z(t,x)\right)g\left(t,z(t,x),I_t^{\gamma_2}z(t,x)\right)\Big\|$$
$$\le \Big\|f\left(t,z_n(t,x),I_t^{\eta-\gamma_1}y_n(t,x)\right)g\left(t,z_n(t,x),I_t^{\gamma_2}z_n(t,x)\right)\Big\| + \Big\|f\left(t,z(t,x),I_t^{\nu-\gamma_1}y(t,x)\right)g\left(t,z(t,x),I_t^{\gamma_2}z(t,x)\right)\Big\|$$
$$\le \Big\|f\left(t,z_n(t,x),I_t^{\eta-\gamma_1}y_n(t,x)\right)\Big\|\,\Big\|g\left(t,z_n(t,x),I_t^{\gamma_2}z_n(t,x)\right)\Big\| + \Big\|f\left(t,z(t,x),I_t^{\nu-\gamma_1}y(t,x)\right)\Big\|\,\Big\|g\left(t,z(t,x),I_t^{\gamma_2}z(t,x)\right)\Big\|$$
$$\le 2K_f(t)\Omega_{\mathrm{f}}\left(\|z\|+\frac{T^{\nu-\gamma_1}}{\Gamma(\nu-\gamma+1)}\|y\|\right) K_g(t)\Omega_{\mathrm{g}}\left(\|z\|+\frac{T^{\gamma_2}}{\Gamma(\gamma_2+1)}\|y\|\right)$$

Therefore,

$$\|Fz_n-Fz\| = \sup_{t\in J}\Big\|\int_0^t \tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}P^{-1} \Big[f\left(s,z_n,\ {}^RD_s^{\gamma_1}z_n(s,x)\right)g\left(s,z_n,I_s^{\gamma_2}z_n(s,x)\right) - f\left(s,z,\ {}^RD_s^{\gamma_1}z(s,x)\right)g\left(s,z,I_s^{\gamma_2}z(s,x)\right)\Big]\,ds\Big\|$$

$$\le \left\|\tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}\right\|\|P^{-1}\| \int_0^t\Big\|f\left(s,z_n,\ {}^RD_s^{\gamma_1}z_n(s,x)\right)g\left(s,z_n,I_s^{\gamma_2}z_n(s,x)\right) -f\left(s,z,\ {}^RD_s^{\gamma_1}z(s,x)\right)g\left(s,z,I_s^{\gamma_2}z(s,x)\right)\Big\|\,ds\to 0$$

and

$$\|\ {}^RD_t^{\nu}(Fz_n)-\ {}^RD_t^{\nu}(Fz)\| = \sup_{t\in J}\Big\|\int_0^t \ {}^RD_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}P^{-1} \Big[f\left(s,z_n,\ {}^RD_s^{\gamma_1}z_n(s,x)\right)g\left(s,z_n,I_s^{\gamma_2}z_n(s,x)\right)- f\left(s,z,\ {}^RD_s^{\gamma_1}z(s,x)\right)g\left(s,z,I_s^{\gamma_2}z(s,x)\right)\Big]ds\Big\|$$
$$\le \left\|\ {}^RD_t^{\nu}\tilde{S}_{\alpha_1+\alpha_2}(t-s)(t-s)^{\alpha_1+\alpha_2-1}\right\|\|P^{-1}\|$$

$$\int_0^t\Big\|f\left(s,z_n(s,x),\ {}^RD_s^{\gamma_1}z_n(s,x)\right)g\left(s,z_n(s,x),I_s^{\gamma_2}z_n(s,x)\right) -f\left(s,z(s,x),\ {}^RD_s^{\gamma_1}z(s,x)\right)g\left(s,z(s,x),I_s^{\gamma_2}z(s,x)\right)\Big\|\,ds\to 0$$

Then F is continuous and from (3.15), (3.16) F is equicontinuous and from (3.17), (3.18) F is precompact this means F is a completely continuous and from (3.14) the set of solutions

$$\left(z_{\alpha_1+\alpha_2},y_{\alpha_1+\alpha_2}\right) = \tau\left(\Phi\left(z_{\alpha_1+\alpha_2},y_{\alpha_1+\alpha_2}\right),\Psi\left(z_{\alpha_1+\alpha_2},y_{\alpha_1+\alpha_2}\right)\right),$$

$0<\tau<1$ is bounded. Then F has fixed point in $C_\nu^{RL}([0,T],X)$ (Schaefer theorem) therefore every fixed point of F is a mild solution of (3.4) on $[0,T]$.

Conclusion

Concluded that the solvability of fractional order intgro-partial differential system needed the Laplace fractional transforms with Mainardi's function for computing the mild solution and the certain semigroup play important role. The assumptions presented in this work are necessary conditions for proving the solvability by transforming the problem to abstract Cauchy problem and the nonlinear functional analysis make a big role with specifying the space and domain of operators to generalize the problem of extensible beam equations and other certain problems.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Al-Mustansiriyah.

References:

1. Abbas M I.Existence and Uniqueness of Mittag-Leffler-Ulam Stable Solution for Fractional Integrodifferential Equations with Nonlocal Initial Conditions. Eur. J. Pure Appl. Math. 2015 Oct. 28;8(4):478-98.
2. Abbas S, Benchohra M, Darwish MA. Some new existence results and stability concepts for fractional partial random differential equations. J. Math. Appl. 2016; 39:5–22.
3. Agarwal RP, Ntouyas S K, Ahmad B, Alhothuali M S. Existence of solutions for integro-differential equations of fractional order with nonlocal three-point fractional boundary conditions. Adv. Differ. Equ. 2013 Dec 1; 2013(1):128.
4. Balachandran K, Park JY, Jung I H. Existence of Solutions of Nonlinear Extensible Beam Equations. Math. Comput. Model. 2002; 36: 747-754.
5. Bazgir H, Ghazanfari B. Existence of Solutions for Fractional Integro-Differential Equations with Non-Local Boundary Conditions. Math. Comput. Appl. 2018 Sep; 23(3):36.
6. Fazli H, Nieto J, Bahrami F. On the existence and uniqueness results for nonlinear sequential fractional differential equations. Appl. Comput. Math. 2018; 17(1):36-47.
7. Engel K J, Nagel R. One Parameter Semigroupfor Linear Evolution Equations. Springer-Verlag, New York, Berlin, 2000.
8. Guezane-L A, Khaldi R. Positive Solutions for Multi-order Nonlinear Fractional Systems. Int. J. Anal. Appl. 2017 Aug 25; 15(1):18-22.
9. Guezane-L A, Ramdane S. Existence of solutions for a system of mixed fractional differential equations. JTUS. 2018 Jul 4; 12(4):421-6.
10. Guo L, Liu L, Wu Y. Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions. Nonlinear Anal. Model. Control. 2016 Jan 1;21(5):635-50.

11. Lakoud A G, Khaldi R, Kılıcman A. Existence of solutions for a mixed fractional boundary value problem. Adv. Differ. Equ. 2017 Dec 1; 2017(1):164.
12. Li K. Stochastic Delay Fractional Evolution Equations Driven by Fractional Brownian Motion. Math. Meth. Appl. Sci. 2015 May 30;38(8):1582-91.
13. Liu Y, Li S, Yang X. Existence and Uniqueness of Positive Solutions for (n−1,1)−Type BVPs of Two-Term Fractional Differential Equations. Progr. Fract. Differ. Appl. 1 Jul. 2016; 2(3):207-217.
14. Pazy, A. Semigroup of Linear Operator and Applications to Partial Differential Equations. Springer-Verlag, New York, 1983.
15. Podlubny I. Fractional Differential Equations. Academic Press, San Diego. California, USA, 1999.
16. Qiao Y, Zhou Z. Existence of positive solutions of singular fractional differential equations with infinite-point boundary conditions. Adv. Differ. Equ. 2017 Dec 1; 2017(1):8.
17. TatarN E. Existence of mild solutions for a neutral fractional equation with fractional nonlocal conditions. Electron. J. Diff. Eq. 2012 Jan 1; 2012(153):1-2.
18. Wang G, Liu S, Baleanu D, Zhang L. Existence results for nonlinear fractional differential equations involving different Riemann-Liouville fractional derivatives. Adv. Differ. Equ. 2013 Dec 1; 2013(1):280.
19. Wang J, Fe˘ckan M, Zhou Y. Presentation of solutions of impulsive fractional Langevin equations and existence results. Eur. Phys. J. Special Topics. 2013 Sep 1; 222(8):1857-74.

قابلية الحل لبعض الأنظمة من المعادلات المتعددة التفاضلية التكاملية ذات الرتب الكسرية

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الخلاصة

خلال البحث تم المناقشة وبالتفصيل قابلية الحل لبعض الأنظمة من المعادلات المتعددة التفاضلية التكاملية ذات الرتب الكسرية باستخدام مفهوم مسائل كوشي المختصرة وايضا نظرية شبه زمرة المختارة مع بعض الشروط الضرورية والكافية.

الكلمات المفتاحية : التفاضل الكسري، الحل المعتل، التحليل الدالي غير الخطي، نظرية النقطة الثابتة، نظرية شبه الزمرة.