Open Access
2020, 17(4):1271-1282

Baghdad Science Journal

P-ISSN: 2078-8665
E-ISSN: 2411-7986

DOI: http://dx.doi.org/10.21123/bsj.2020.17.4.1271

The Numerical Technique Based on Shifted Jacobi-Gauss-Lobatto Polynomials for Solving Two Dimensional Multi-Space Fractional Bioheat Equations

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Received 26/7/2019, Accepted 18/2/2020, Published 1/12/2020

Abstract:

This article deals with the approximate algorithm for two dimensional multi-space fractional bioheat equations (M-SFBHE). The application of the collection method will be expanding for presenting a numerical technique for solving M-SFBHE based on "shifted Jacobi-Gauss-Labatto polynomials" (SJ-GL-Ps) in the matrix form. The Caputo formula has been utilized to approximate the fractional derivative and to demonstrate its usefulness and accuracy, the proposed methodology was applied in two examples. The numerical results revealed that the used approach is very effective and gives high accuracy and good convergence.

Key words: Accuracy, Collocation method, Shifted Jacobi-Gauss-Lobatto polynomials, Two dimensional fractional bioheat equation.

Introduction

Fractional calculus has been utilized to ameliorate the modeling fineness of many phenomena naturalistic in science and engineering. It was applied in the assorted fields such as diffusion problems, viscoelasticity, mechanics of solids, biomedical engineering, control theory, and economics, etc. (1).

Pennes' suggested in (1948) the essential structure of the mathematical designing that describes temperature propagation in human tissues, the model known as the bioheat equation remains extensively used in the hyperthermal and freezing treatments (2). The fractional bioheat model which extracted the focus of the researchers and these contributed to a significant amount of the researches based on approximate and analytic methodology, for example (Singh et al. in (3), finite difference and homotopy perturbation method, Jiang and Qi in (4), Taylor's series expansion, Damor et al. in (5), implicit finite difference method, Ezzat et al. in (6), Laplace transform mode, Ferrás et al. and Kumar et al. in (7-8), implicit finite difference method, "backward finite difference method" and "Legendre wavelet Galerkin scheme", Qin and Wu and Damor et al. in (9-10), quadratic spline collocation method and Fourier-Laplace transforms, Kumar and Rai in (11), finite element based on Legendre wavelet Galerkin method, Roohi et al. in (12), Galerkin scheme, Hosseininia et al. and Al-Saadawi and Al-Humedi in (13-14), "Legendre wavelet method" and "Collocation method").

In this paper, the SJ-GL-Ps in the matrix form is employed for the present numerical approach in order to solving the following two-dimensional M-SFBHE. Therefore, the space-fractional version of the two-dimensional unsteady state Pennes bioheat equation can be obtained by replacing the space derivatives with the derivatives of arbitrary positive real orders $v_1, v_2 \in (1,2]$ as:

$$\rho c \frac{\partial T(x,y,t)}{\partial t} - K\left(\frac{\partial^{v_1} T(x,y,t)}{\partial x^{v_1}} + \frac{\partial^{v_2} T(x,y,t)}{\partial y^{v_2}}\right) + W_b c_b (T(x,y,t) - T_a) = Q_{ext}(x,y,t) + Q_{met}, \quad 0 \le t \le \mathcal{T}, 0 \le x \le R_1, 0 \le y \le R_2, \quad \dots (1)$$

with initial and boundary conditions

$$T(x,y,0) = T_c, 0 < x < R_1, 0 < y < R_2, \quad \dots (2)$$

$$-K\frac{\partial T(0,y,t)}{\partial x} = q_0, 0 \le y \le R_2, t > 0, \quad \dots (3)$$

$$-K\frac{\partial T(x,0,t)}{\partial y}=q_1, 0\le x\le R_1, t>0, \quad \dots(4)$$
$$-K\frac{\partial T(R_1,y,t)}{\partial x}=0, 0\le y\le R_2, t>0, \quad \dots(5)$$
$$-K\frac{\partial T(x,R_2,t)}{\partial y}=0, 0\le x\le R_1, t>0, \quad \dots(6)$$
where $\rho, c, K, T, t, x, y, T_a, q_0, W_b=\rho_b w_b, Q_{ext}$ and Q_{met} symbolizes density, specific heat, thermal conductivity, temperature, time, distances with x, y, artillery temperature, heat flux on the skin surface, blood exudation rate, metabolic heat obstetrics in lacing tissue and external heat exporter in skin tissue respectively. The units and value of the symbolizations that expressed in this equation are tabulating in Table1.

Table 1(15)**.** The units and values utilized in this paper of the M-SFBHE.

Symbols	T_a	ρ , ρ_b	c, c_b	K	ω_b	Q_{met}
Units	°C	kg/m^3	J/kg°C	W/m°C	m^3/s/m^3	W/m^3
values	37	1000	4000	0.5	0.0005	420

The sections of this article are structured as follows: In the next section, some definitions of essentials principles of the fractional calculus will be showing. Followed by the shifted Jacobi polynomials operational matrix for ordinary derivatives and their fractional derivatives, the approximate approach for 1D, 2D and 3D temperature function in matrix form depending on shifted Jacobi polynomials for fractional differentiation are given, to establish a numerical solution for M-SFBHE, so a method for solution is explained, after that to determine an error bound $T(x,y,t)$ is called for, an efficient error estimation for the SJ-GL-Ps will be given. The final section deals with the numerical results for the M-SFBHE.

Preliminaries and Notations

The essentials principles of the fractional calculus theory that utilized in this article will be explain.

Definition (16): The Riemann-Liouville fractional integral of order $v>0$ defined as:
$$I^v\emptyset(x)=\frac{1}{\Gamma(v)}\int_0^x (x-s)^{v-1}\emptyset(s)ds, v>0, \quad \dots(7)$$
$$I^0\emptyset(x)=\emptyset(x).$$
Definition (12): The Riemann-Liouville definition of fractional differential operator where $v>0$ given as follows:
$$D^v\emptyset(x)=\begin{cases}\dfrac{1}{\Gamma(n-v)}\dfrac{d^n}{dx^n}\displaystyle\int_0^x\frac{\emptyset(s)}{(x-s)^{v-n+1}}ds, n-1\le v<n,\\ \dfrac{d^n\emptyset(x)}{dx^n} \qquad , v=n.\end{cases} \quad \dots(8)$$

Definition (11): The Caputo definition of fractional differential operator defined as:
$$D^v\emptyset(x)=\begin{cases}\dfrac{1}{\Gamma(n-v)}\displaystyle\int_0^x\frac{\emptyset^{(n)}(s)}{(x-s)^{v-n+1}}ds, n-1\le v<n,\\ \dfrac{d^n\emptyset(x)}{dx^n} \qquad , v=n.\end{cases} \quad \dots(9)$$
The relation that governing the Riemann-Liouville and Caputo of fractional order given via the forms (3):
$$D^vI^v\emptyset(x)=\emptyset(x),$$
$$I^vD^v\emptyset(x)=\emptyset(x)-\sum_{k=0}^{n-1}\emptyset^{(k)}(0^+)\frac{x^k}{k!}. \quad \dots(10)$$
For $\beta\ge 0, v\ge -1$, and constant C, Caputo fractional derivative has some fundamental properties which are needed here as follows (17):

$i)\ D^vC=0,$

$$ii)\ D^v x^\beta=\begin{cases}0 & \text{for } \beta\in\mathbb{N}_0 \text{ and } \beta<\lceil v\rceil\\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-v+1)}x^{\beta-v}, & \beta\in or\notin\mathbb{N}_0, \beta\ge\lceil v\rceil,\end{cases} \quad \dots(11)$$
$$iii)\ D^v\left(\sum_{i=0}^n c_i\emptyset_i(x)\right)=\sum_{i=0}^n c_iD^v\emptyset_i(x),$$
where $\{c_i\}_{i=0}^n$ are constant.

Definition (17): (generalized Taylor's formula). Assume that $D^{iv}\emptyset(t)\in\mathbb{C}(0,1)$ for $i=0(1)(n-1)$, then one has:
$$\emptyset(x)=\sum_{i=0}^{n-1}\frac{x^{iv}}{\Gamma(iv+1)}D^{iv}\emptyset(0^+)+\frac{x^{nv}}{\Gamma(nv+1)}D^{nv}\emptyset(\xi) \quad \dots(12)$$
where $0<\xi\le x$, $\forall x\in(0,R)$. Also, one has assume
$$\left|\emptyset(x)-\sum_{i=0}^{n-1}\frac{x^{iv}}{\Gamma(iv+1)}D^{iv}\emptyset(0^+)\right|\le M_v\frac{x^{nv}}{\Gamma(nv+1)} \quad \dots(13)$$
and $M_v\ge|D^{nv}\emptyset(\xi)|$.

In case $v=1$, the generalized Taylor's formula in Eq. (10) is the classical Taylors formula.

Shifted Jacobi Polynomials for Ordinary Derivatives and Fractional Derivatives

The Jacobi polynomials which are orthogonal in the interval $[-1,1]$ are defined as the following formula:
$$P_i^{(\alpha,\beta)}(t)=\frac{(i+\alpha+\beta-1)\{\alpha^2-\beta^2+t(2i+\alpha+\beta)(2i+\alpha+\beta-2)\}}{2i(i+\alpha+\beta)(2i+\alpha+\beta-2)}P_{i-1}^{(\alpha,\beta)}(t)-\frac{(i+\alpha-1)(i+\beta-1)(2i+\alpha+\beta)}{i(i+\alpha+\beta)(2i+\alpha+\beta-2)}P_{i-2}^{(\alpha,\beta)}(t),$$
$$n=2,3,\dots \quad \dots(14)$$

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2020, 17(4):1271-1282 | E-ISSN: 2411-7986

where $P_0^{(\alpha,\beta)}(t)=1$, $P_1^{(\alpha,\beta)}(t)=\frac{1}{2}[(\alpha+\beta+2)t+(\alpha-\beta)]$.

For transform Jacobi polynomials on a region $0\le x\le R$, one can procedure the replace of variables $t=\frac{2x}{R}-1$ in the above formula. Therefore, the shifted Jacobi polynomials (SJPs) are constructed in the relation as follows (2)

$$P_i^{(\alpha,\beta)}\left(\frac{2x}{R}-1\right)=P_{R,i}^{(\alpha,\beta)}(x)\ ,x\in[0,R].$$

The analytic form of the SJPs $P_{R,i}^{(\alpha,\beta)}(x)$ of degree i is given as following:

$$P_{R,i}^{(\alpha,\beta)}(x)=\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+k+\alpha+\beta+1)\Gamma(i+\beta+1)}{R^k\Gamma(i+\alpha+\beta+1)\Gamma(k+\beta+1)(i-k)!\,k!}x^k\ ,i\in N, \quad \dots(15)$$

where $P_{R,i}^{(\alpha,\beta)}(0)=(-1)^i\frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)i!}$, $P_{R,i}^{(\alpha,\beta)}(R)=\frac{\Gamma(\alpha+i+1)}{i!\Gamma(\alpha+1)}$.

From the SJPs, the formulas that most utilized can be obtain which are the "shifted Legendre polynomials" (SLPs) $L_i(x)$; the "shifted Chebyshev polynomials" (SCPs) of the first kind $T_{R,i}(x)$; the "shifted Chebyshev polynomials" of the second kind $U_{R,i}(x)$; the nonsymmetric SJPs, the two important special cases of "shifted Chebychev polynomials" of third(fourth) kinds $V_{R,i}(x)$ and $W_{R,i}(x)$; and also, the symmetric SJPs that called "Gegenbauer (ultraspherical) polynomials" $C_{R,i}^{\alpha}(x)$.These orthogonal polynomials are interrelated to the SJPs by the following relations (18)

$$L_i(x)=P_{R,i}^{(0,0)}(x),\qquad T_{R,i}(x)=\frac{i!\Gamma(\frac{1}{2})}{\Gamma(i+\frac{1}{2})}P_{R,i}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)$$

$$U_{R,i}(x)=\frac{(i+1)!\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i+\frac{3}{2}\right)}P_{R,i}^{\left(\frac{1}{2},\frac{1}{2}\right)}(x),\ V_{R,i}(x)=\frac{(2i)!!}{(2i-1)!!}P_{R,i}^{\left(\frac{1}{2},\frac{-1}{2}\right)}(x),$$

$$W_{R,i}(x)=\frac{(2i)!!}{(2i-1)!!}P_{R,i}^{\left(-\frac{1}{2},\frac{1}{2}\right)}(x),\ C_{R,i}^{\alpha}(x)=\frac{i!\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(i+\alpha+\frac{1}{2}\right)}P_{R,i}^{\left(\alpha-\frac{1}{2},\beta-\frac{1}{2}\right)}(x). \quad \dots(16)$$

The orthogonal property of SJPs is given by

$$\int_0^R P_{R,i}^{(\alpha,\beta)}(x)P_{R,k}^{(\alpha,\beta)}(x)\omega_{R,i}^{(\alpha,\beta)}dx=h_{R,k}, \quad \dots(17)$$

where $\omega_{R,i}^{(\alpha,\beta)}=x^{\beta}(R-x)^{\alpha}$ and

$$h_{R,k}=\begin{cases}\frac{R^{\alpha+\beta+1}}{k!(2k+\alpha+\beta+1)}\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)}, & i=k,\\ 0, & i\neq k.\end{cases}$$

The first-order derivative of the vector $\emptyset(x)=\left[P_{R,0}^{(\alpha,\beta)}(x),P_{R,1}^{(\alpha,\beta)}(x),\dots,P_{R,N}^{(\alpha,\beta)}(x)\right]'$ can be expressed by

$$\frac{d\emptyset(x)}{dx}=D^{(1)}\emptyset(x), \quad \dots(18)$$

where $D^{(1)}$ is the $(N+1)\times(N+1)$ shifted Jacobi operational matrix of derivative introduced by (19):

$$D^{(1)}=d_{ij}=\begin{cases}A_1(i,j), & i>j,\\ 0 & otherwise,\end{cases}\ \text{and}$$

$$A_1(i,j)=\frac{R^{\alpha+\beta}(i+\alpha+\beta+1)(i+\alpha+\beta+2)_j(j+\alpha+2)_{i-j-1}\Gamma(j+\alpha+\beta-1)}{\Gamma(2j+\alpha+\beta+1)(i-j-1)!}\times{}_3F_2\left(\begin{matrix}j-i+1,\ j+i+\alpha+\beta+2, j+i+\alpha+\beta+2, j+\alpha+1\\ j+\alpha+2,\ 2j+\alpha+\beta+2\end{matrix};1\right)$$

For example, for even N we have

$$D^{(1)}=\begin{bmatrix}0&0&0&\cdots&0&0\\ A_1(1,0)&0&0&\vdots&0&0\\ A_1(2,0)&A_1(2,1)&0&\vdots&0&0\\ A_1(3,0)&A_1(3,1)&A_1(3,2)&\vdots&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ A_1(N,0)&A_1(N,1)&A_1(N,3)&\cdots&A_1(N,N-1)&0\end{bmatrix}.$$

To generalize the shifted Jacobi operational matrix of ordinary derivatives into the fractional derivative. By utilizing Eq. (18), it is obvious that

$$\frac{d^n\emptyset(x)}{dx^n}=\left(D^{(1)}\right)^n\emptyset(x), \quad \dots(19)$$

where $n\in N$ and the superscript in $D^{(1)}$, symbolizes matrix powers. Thus

$$D^{(n)}=\left(D^{(1)}\right)^n,\qquad n=1,2. \quad \dots(20)$$

Corollary (1): In the case of $\alpha=\beta=0$, it is obvious that the SJPs for derivatives in the matrix form for integer calculus is in complete agreement with the SLPs for derivatives in the matrix form for integer calculus.

Corollary (19): In the case of $\alpha=\beta=-\frac{1}{2}$, it is clear that the SJPs for derivatives in the matrix form for integer calculus is in complete agreement with the SCPs for derivatives in the matrix form for integer calculus.

Lemma:- Let $P_{R,i}^{(\alpha,\beta)}(x)$ be the SJPs. Then $D^{v}P_{R,i}^{(\alpha,\beta)}(x)=0, i=0,1,2,\dots,\lceil v\rceil-1, v>0$.

Proof:- Using the properties (ii) and (iii) of the Eq. (11) into Eq. (15) lead us to prove the lemma. □

The following theorem is generalizing the operational matrix of derivatives form an arbitrary fractional order based on SJPs that have given in Eq.(18)

Theorem(18):- Suppose that $\emptyset(x)$ be shifted Jacobi vector defined in $\emptyset(x)=\left[P_{R,0}^{(\alpha,\beta)}(x),P_{R,1}^{(\alpha,\beta)}(x),\dots,P_{R,N}^{(\alpha,\beta)}(x)\right]'$ and assume also, $v>0$. Then

$$D^{v}\emptyset(x)\cong D_R^{(v)}\emptyset(x), \quad \dots(21)$$

Baghdad Science Journal
P-ISSN: 2078-8665
E-ISSN: 2411-7986

where $D_R^{(v)}$ is the $(N+1)\times(N+1)$ shifted Jacobi operational matrix of derivatives of fractional order v in the Caputo formula and is defined by:

$$D_R^{(v)} = \begin{bmatrix} & 0 \quad 0 \quad 0 & & \cdots & 0 \\ & \vdots \quad \vdots \quad \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \mathcal{B}^v(\lceil v\rceil,0) & \mathcal{B}^v(\lceil v\rceil,1) & \mathcal{B}^v(\lceil v\rceil,2) & \cdots & \mathcal{B}^v(\lceil v\rceil,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathcal{B}^v(i,0) & \mathcal{B}^v(i,1) & \mathcal{B}^v(i,2) & \cdots & \mathcal{B}^v(i,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathcal{B}^v(N,0) & \mathcal{B}^v(N,1) & \mathcal{B}^v(N,2) & \cdots & \mathcal{B}^v(N,N) \end{bmatrix},$$

where

$$\mathcal{B}^v(i,j) = \sum_{k=\lceil v\rceil}^{i} \delta_{ijk}, \qquad \dots(22)$$

and δ_{ijk} is given by

$$\delta_{ijk} = \frac{(-1)^{i-k}R^{\alpha+\beta-v+1}\Gamma(j+\beta+1)\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{h_{R,j}\Gamma(j+\alpha+\beta+1)\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(k-v+1)(i-k)!} \times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l+\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(l+k+\beta-v+1)}{\Gamma(l+\beta+1)\Gamma(l+k+\alpha+\beta-v+2)(j-l)!\,l!} \quad \dots(23)$$

Proof. By apply Eq. (11) into Eq. (15) (the SJPs $P_{R,i}^{(\alpha,\beta)}(x)$ of degree), the resulting:

$$D^v P_{R,i}^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)D^v x^k}{R^k k!\,\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!}$$
$$= \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)x^{k-v}}{R^k\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!\,\Gamma(k-v+1)}, i = \lceil v\rceil, \lceil v\rceil+1, \dots. \qquad \dots(24)$$

Now, approximate x^{k-v} by $(N+1)$ terms of shifted Jacobi series, the obtained result:

$$x^{k-v} \approx \sum_{j=0}^{N} \mu_{kj}\, P_{R,j}^{(\alpha,\beta)}(x), \qquad \dots(25)$$

where the coefficients μ_{kj} can be obtain as following

$$\mu_{kj} = \frac{1}{h_{R,j}} \int_0^R x^{k-v}\, P_{R,j}^{(\alpha,\beta)}(x) x^\beta (R-x)^\alpha dx$$
$$= \frac{1}{h_{R,j}} \int_0^R x^{k-v}\, x^\beta (R-x)^\alpha \times \sum_{l=0}^{j} (-1)^{j-l} \frac{\Gamma(j+\beta+1)\Gamma(j+l+\alpha+\beta+1)x^l dx}{R^l\Gamma(l+\beta+1)\Gamma(j+\alpha+\beta+1)(j-l)!\,l!}$$
$$= \frac{1}{h_{R,j}} \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+\beta+1)\Gamma(j+l+\alpha+\beta+1)}{R^l\Gamma(l+\beta+1)\Gamma(j+\alpha+\beta+1)(j-l)!\,l!} \times \int_0^R x^{\beta+l+k-v}\,(R-x)^\alpha dx \qquad \dots(26)$$
$$= \frac{1}{h_{R,j}} \sum_{l=0}^{j} \left[(-1)^{j-l} \frac{\Gamma(j+\beta+1)\Gamma(j+l+\alpha+\beta+1)}{R^l\Gamma(l+\beta+1)\Gamma(j+\alpha+\beta+1)(j-l)!\,l!} \times \frac{\Gamma(\alpha+1)\Gamma(\beta+l+k-v+1)}{\Gamma(\beta+l+k-v+2)} R^{\alpha+\beta+k+l-v+1}\right]$$
$$= \frac{\Gamma(j+\beta+1)R^{\alpha+\beta+k-v+1}}{\Gamma(j+\alpha+\beta+1)h_{R,i}} \times \sum_{l=0}^{j} (-1)^{j-l} \frac{\Gamma(\alpha+1)\Gamma(j+l+\alpha+\beta+1)\Gamma(\beta+l+k-v+1)}{\Gamma(l+\beta+1)\Gamma(\alpha+\beta+l+k-v+2)(j-l)!\,l!}.$$

Now, substituting Eq. (25) into Eq. (23), observe that

$$D^v P_{R,i}^{(\alpha,\beta)}(x) = \sum_{j=0}^{N} \mathcal{B}^v(i,j) P_{R,j}^{(\alpha,\beta)}(x),$$
$$i = \lceil v\rceil, \lceil v\rceil+1, \dots, N \qquad \dots(27)$$

where $\mathcal{B}^v(i,j)$ is given in Eq. (21).

The Eq. (26) in a vector form can be write as:

$$D^v P_{R,i}^{(\alpha,\beta)}(x) \approx [\mathcal{B}^v(i,0), \mathcal{B}^v(i,1), \dots, \mathcal{B}^v(i,N)]\emptyset(x), i = \lceil v\rceil(1)N. \qquad \dots(28)$$

Also from above Lemma, the obtained equation:

$$D^v P_{R,i}^{(\alpha,\beta)}(x) \approx [0,0,\dots,0]\emptyset(x), i = 0(1)(\lceil v\rceil-1), v>0. \qquad \dots(29)$$

By a combination of the Eqs. (28-29), the desired result will obtain.

One can notes that if $v = n \in N$, then above theorem gives the same formula as in Eq. (18).

Corollary (1):- If $\alpha = \beta = 0$ and $R = 1$, then δ_{ijk} is given as follows:

$$\delta_{ijk} = \frac{(-1)^{i-k}\Gamma(j+1)\Gamma(i+1)\Gamma(i+k+1)}{h_{R,j}(i-k)!\,\Gamma(j+1)\Gamma(k+1)\Gamma(i+1)\Gamma(k-v+1)} \times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l+1)\Gamma(l+k-v+1)}{(j-l)!\,l!\,\Gamma(l+1)\Gamma(l+k-v+2)}.$$

By the aid of properties of the SJPs with simplification, the obtained result is:

$$\delta_{ijk} = \varphi_{ijk} = (2j+1) \sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(j+l)!\,(i+k)!}{(l+k-v+1)(i-k)!\,k!\,(j-l)!\,(l!)^2\Gamma(k-v+1)}.$$

Then one can easily demonstrated that

$$\mathcal{B}^v(i,j) = \sum_{k=\lceil v\rceil}^{i} \varphi_{ijk},$$

where φ_{ijk} is given as in (1). It is clear that the SJPs for derivatives in the matrix form for fractional calculus with $\alpha = \beta = 0$, is in complete agreement with the SLPs for derivatives in the matrix form for fractional calculus as in (1).

Corollary (20):- If $\alpha = \beta = -\frac{1}{2}$ then δ_{ijk} given as follows:

$$\delta_{ijk} = \frac{(-1)^{i-k}\mathrm{R}^{-v}\Gamma(j+\frac{1}{2})\Gamma(i+\frac{1}{2})\Gamma(i+k)}{\epsilon_{R,j}\Gamma(j)\Gamma(\frac{1}{2}+k)\Gamma(i)\Gamma(k-v+1)(i-k)!} \times \sum_{l=0}^{j}\frac{(-1)^{j-l}\Gamma(j+l)\Gamma(l+k-v+\frac{1}{2})}{\Gamma(l+\frac{1}{2})\Gamma(l+k-v+1)(j-l)!\,l!},$$

by the aid of properties of the SJPs with simplification, the obtained result is:

$$\delta_{ijk} = \varphi_{ijk} = \frac{(-1)^{i-k}2i(i+k-1)!\,\Gamma(k-v+\frac{1}{2})}{\epsilon_{R,j}\mathrm{R}^{v}\Gamma(k+\frac{1}{2})(i-k)!\,\Gamma(k-v-j+1)\Gamma(k+j-v+1)}, j = 0(1)N.$$

Then one can easily elucidate that

$$\mathcal{B}^{v}(i,j) = \sum_{k=\lceil v\rceil}^{i}\varphi_{ijk},$$

where φ_{ijk} and $\epsilon_{R,j}$ are given as in (20). It is clear that the SJPs for derivatives in the matrix form for any arbitrary fractional order with $\alpha=\beta=-\frac{1}{2}$, is complete accord with the SCPs for derivatives in the matrix form for fractional calculus obtained by (20).

Shifted Jacobi Operational Matrix of Fractional Differentiation

A temperature function $T(x)$ define for $0 \le x \le R_1$ may be expressed in terms of the SJPs as

$$T(x) = \sum_{i=0}^{\infty} c_i\, P_{R_1,i}^{(\alpha,\beta)}(x), \quad \dots(30)$$

where the coefficients c_i are given by

$$c_i = \frac{1}{h_{R_1,i}}\int_0^{R_1} T(x)\, P_{R_1,i}^{(\alpha,\beta)}(x)\omega_{R_1,i}^{(\alpha,\beta)}(x)dx\,, i = 0,1,2,\dots. \quad \dots(31)$$

In practice, consider the $(N+1)$-term SJPs so that

$$T(x) \approx \sum_{i=0}^{N} c_i\, P_{R_1,i}^{(\alpha,\beta)}(x) = C'\emptyset(x), \quad \dots(32)$$

where the shifted Jacobi coefficient vector c_i and the shifted Jacobi vector $\emptyset(x)$ are given by $C' = [c_0, c_1, \dots, c_N], \emptyset(x) = \left[P_{R_1,0}^{(\alpha,\beta)}(x), P_{R_1,1}^{(\alpha,\beta)}(x), \dots, P_{R_1,N}^{(\alpha,\beta)}(x)\right]'$.

By extending the above property in two variable functions, can approximate a two variable function $T(x,y)$ define for $0 \le x \le R_1$ and $0 \le t \le \mathcal{T}$ dependent on double SJPs as

$$T(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\, P_{R_1,i}^{(\alpha,\beta)}(x)P_{\mathcal{T},j}^{(\alpha,\beta)}(t), \quad \dots(33)$$

where

$$a_{ij} = \frac{1}{h_{R,i}h_{\mathcal{T},j}}\int_0^{R_1}\int_0^{\mathcal{T}} T(x,t)\, P_{R_1,i}^{(\alpha,\beta)}(x)P_{\mathcal{T},j}^{(\alpha,\beta)}(t)\omega^{(\alpha,\beta)}(x,t)dtdx, \quad \dots(34)$$

such that $\omega^{(\alpha,\beta)}(x,t) = \omega_{R_1,i}^{(\alpha,\beta)}(x)\, \omega_{\mathcal{T},j}^{(\alpha,\beta)}(t)$.

In practice, consider the $(N+1)$ and $(M+1)$-terms double SJPs with respect to x, t so that

$$T_{N,M}(x,t) \approx \sum_{i=0}^{N}\sum_{j=0}^{M} a_{ij}\, P_{R_1,i}^{(\alpha,\beta)}(x)P_{\mathcal{T},j}^{(\alpha,\beta)}(t) = \emptyset(x)'A\,\emptyset(t), \quad \dots(35)$$

where the shifted Jacobi coefficient matrix A and the shifted Jacobi vectors $\emptyset(x)$ and $\emptyset(t)$ are given by:

$$A = \{a_{ij}\}_{i,j=0}^{N,M},$$
$$\emptyset(x) = \left[P_{R_1,0}^{(\alpha,\beta)}(x), P_{R_1,1}^{(\alpha,\beta)}(x), \dots, P_{R_1,N}^{(\alpha,\beta)}(x)\right]',$$
$$\emptyset(t) = \left[P_{\mathcal{T},0}^{(\alpha,\beta)}(t), P_{\mathcal{T},1}^{(\alpha,\beta)}(t), \dots, P_{\mathcal{T},M}^{(\alpha,\beta)}(t)\right]'.$$

Now, in order to approximate a three variable temperature function $T(x,y,t)$ define for $0 \le x \le R_1, 0 \le y \le R_2$ and $0 \le t \le \mathcal{T}$ dependent on triple Jacobi series as

$$T(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tilde{t}_{kij}P_{R_1,i}^{(\alpha,\beta)}(x)P_{R_2,i}^{(\alpha,\beta)}(y)P_{\mathcal{T},j}^{(\alpha,\beta)}(t) \quad \dots(36)$$

In practice, consider the $(N_1+1), (N_2+1)$ and $(M+1)$-terms triple SJPs with respect to x, y, t so that where

$$\tilde{t}_{kij} = \int_0^{R_1}\int_0^{R_2}\frac{\int_0^{\mathcal{T}} T(x,y,t)\, P_{R_1,i}^{(\alpha,\beta)}(x)P_{R_2,i}^{(\alpha,\beta)}(y)P_{\mathcal{T},j}^{(\alpha,\beta)}(t)\omega^{(\alpha,\beta)}(x,y,t)}{h_{R_1,i}h_{R_2,j}h_{\mathcal{T},k}}dtdydx \quad \dots(37)$$

such that

$$\omega^{(\alpha,\beta)}(x,y,t) = \omega_{R_1,i}^{(\alpha,\beta)}(x)\;\omega_{R_2,j}^{(\alpha,\beta)}(y)\omega_{\mathcal{T},k}^{(\alpha,\beta)}(t).$$

$$T_{M,N_1,N_2}(x,y,t) \approx \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{M} \tilde{t}_{kij}P_{R_1,i}^{(\alpha,\beta)}(x)P_{R_2,j}^{(\alpha,\beta)}(y)P_{\mathcal{T},k}^{(\alpha,\beta)}(t) = \emptyset(t)'\,\ddot{T}\,\emptyset(x)\otimes\emptyset(y), \quad \dots(38)$$

where the symbol $\otimes$ is the Kronecker tensor product, the shifted Jacobi vectors $\emptyset(x)$, $\emptyset(y)$ and $\emptyset(t)$ are given by

$$\left.\begin{aligned}\emptyset(x) &= \left[P_{R_1,0}^{(\alpha,\beta)}(x), P_{R_1,1}^{(\alpha,\beta)}(x), \dots, P_{R_1,N_1}^{(\alpha,\beta)}(x)\right]'\\ \emptyset(y) &= \left[P_{R_2,0}^{(\alpha,\beta)}(y), P_{R_2,1}^{(\alpha,\beta)}(y), \dots, P_{R_2,N_2}^{(\alpha,\beta)}(y)\right]'\\ \emptyset(t) &= \left[P_{\mathcal{T},0}^{(\alpha,\beta)}(t), P_{\mathcal{T},1}^{(\alpha,\beta)}(t), \dots, P_{\mathcal{T},M}^{(\alpha,\beta)}(t)\right]'\end{aligned}\right\} \quad \dots(39)$$

Also shifted Jacobi coefficient matrix $\ddot{T}$ is given in a block form as follows

$$\ddot{T} = \begin{bmatrix} \tilde{t}_{000} & \tilde{t}_{001} & \cdots & \tilde{t}_{00N_2} & \tilde{t}_{010} & \tilde{t}_{011} & \cdots & \tilde{t}_{0N_1N_2} \\ \tilde{t}_{100} & \tilde{t}_{101} & \cdots & \tilde{t}_{10N_2} & \tilde{t}_{110} & \tilde{t}_{111} & \cdots & \tilde{t}_{1N_1N_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{t}_{M00} & \tilde{t}_{M01} & \cdots & \tilde{t}_{M0N_2} & \tilde{t}_{M10} & \tilde{t}_{M11} & \cdots & \tilde{t}_{MN_1N_2} \end{bmatrix} \quad \dots(40)$$

Method for Solution

The selection of collocation points is playing significant role in the efficiency and convergence of the "collocation method". For boundary value problems, the "Gauss-Lobatto" points represent one of the principal keys utilized for approximation. It should be renowned that for a differential equation with the singularity at $x = 0$ in the region $[0,\ R]$ one is unable to apply the "collocation method" with "Jacobi-Gauss-Lobatto" points because the two assigned abscissas 0 and R are necessary to use as two points from the collocation nodes. Use the "collocation method" with "Jacobi-Gauss-Lobatto" nodes to treat the two dimensional M-FSBHE; i.e., collocate this equation only at the $M \times (N_1 - 1) \times (N_2 - 1)$ "Jacobi-Gauss-Lobatto" points $(0,\ \mathcal{T}), (0,\ R_1)$ and $(0,\ R_2)$ respectively. These equations and with initial, boundary conditions generate $(M + 1) \times (N_1 + 1) \times (N_2 + 1)$ nonlinear algebraic equations by using one of the iteration methods can be solved.

Now, set $\boldsymbol{P}_{N_1}(0, R_1) = span\left\{P_{R_1,0}^{(\alpha,\beta)}(x), P_{R_1,1}^{(\alpha,\beta)}(x), \dots, P_{R_1,N_1}^{(\alpha,\beta)}(x)\right\}$.

Recall the "Jacobi-Gauss-Lobatto" generators. Such that N_1 is any positive integer, $\boldsymbol{P}_{N_1}(0, R_1)$ stands for the group of all algebraic polynomials from degree at most N_1. Denoting $x_{N_1,i}$ by $x_{R_1,N_1,i}$, and $\omega_{N_1,i}^{(\alpha,\beta)}$ by $\omega_{R_1,N_1,i}^{(\alpha,\beta)}$, $0 \le i \le N_1$, to the grid points and "Ghristoffel numbers" (19) of the standard or "shifted Jacobi-Gauss-Lobatto" quadrature on the $(-1,1)$ or $(0, R_1)$ respectively.

$$x_{R_1,N_1,i} = \frac{R_1}{2}\left(x_{N_1,i} + 1\right), \qquad 0 \le i \le N_1, \quad \dots (41)$$

$$\omega_{R_1,N_1,i}^{(\alpha,\beta)} = \left(\frac{R_1}{2}\right)^{\alpha+\beta+1} \omega_{R_1,i}^{(\alpha,\beta)}.$$

For any $\emptyset(x) \in \boldsymbol{P}_{N_1}(0, R_1)$, we have

$$\int_0^R \omega_{R_1,i}^{(\alpha,\beta)} \emptyset(x)dx = \left(\frac{R_1}{2}\right)^{\alpha+\beta+1} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P\left(\frac{R_1}{2}(x+1)\right) = \left(\frac{R_1}{2}\right)^{\alpha+\beta+1} \sum_{i=0}^{N_1} \omega_{R_1,i}^{(\alpha,\beta)}(x_{N_1,i})\, \emptyset\left(\frac{R_1}{2}\left(x_{N_1,i} + 1\right)\right) = \sum_{i=0}^{N_1} \omega_{R_1,N_1,i}^{(\alpha,\beta)} \emptyset\left(x_{R_1,N_1,i}\right), \quad \dots (42)$$

where $x_{R_1,N_1,i}$ and $\omega_{R_1,N_1,i}^{(\alpha,\beta)}$ are the grid points and equivalent weights of the "shifted Jacobi-Gauss-quadrature" technique on the region $[0, R_1]$ respectively. In the same procedure on the intervals $[0,\ R_2]$ and $[0,\ \mathcal{T}]$ then one can readily show that

$$y_{R_2,N_2,j} = \frac{R_2}{2}\left(y_{N_2,j} + 1\right), \qquad 0 \le j \le N_2, \quad \dots (43)$$

$$t_{\mathcal{T},M,k} = \frac{\mathcal{T}}{2}\left(t_{M,k} + 1\right), \qquad 0 \le k \le M, \qquad \dots (44)$$

$\omega_{R_2,N_2,j}^{(\alpha,\beta)} = \left(\frac{R_2}{2}\right)^{\alpha+\beta+1} \omega_{R_2,j}^{(\alpha,\beta)}$ and $\omega_{\mathcal{T},M,k}^{(\alpha,\beta)} = \left(\frac{\mathcal{T}}{2}\right)^{\alpha+\beta+1} \omega_{\mathcal{T},k}^{(\alpha,\beta)}$.

Now, the proposed method algorithm for Eq. (1) will build based on SJ-GL-Ps, under the given conditions, in the series or matrix form by utilizing Eqs. (41, 43-44) into the shifted Jacobi vectors $\emptyset(x)$, $\emptyset(y)$ and $\emptyset(t)$ define by Eq. (39). In addition, the "shifted Jacobi-Gauss-Labatto" coefficient matrix $\ddot{T}$ is given by Eq. (40).

The 1st temporal and spatial derivatives and their fractional derivatives can approximate as:

$$\left.\begin{aligned} \frac{\partial T(x,y,t)}{\partial t} &= \left[D_{\mathcal{T}}^{(1)} \emptyset(t)\right]' \ddot{T} \emptyset(x) \otimes \emptyset(y), \\ \frac{\partial^{v_1} T(x,y,t)}{\partial x^{v_1}} &= \emptyset'(t) \ddot{T} \left[D_{R_1}^{(v_1)} \emptyset(x)\right] \otimes \emptyset(y), \\ \frac{\partial^{v_2} T(x,y,t)}{\partial y^{v_2}} &= \emptyset'(t) \ddot{T} \emptyset(x) \otimes \left[D_{R_2}^{(v_2)} \emptyset(y)\right] \end{aligned}\right\} \dots (45)$$

The solution method for the two dimensional M-SFBHE can be applied based on Jacobi-Gauss-Labatto in the matrix form that given in Eq. (1), the resulted function will be:

$$\rho c \emptyset'(t) \left[D_{\mathcal{T}}^{(1)}\right]' \ddot{T} \emptyset(x) \otimes \emptyset(y) - K\left(\emptyset'(t) \ddot{T} \left[D_{R_1}^{(v_1)} \emptyset(x)\right] \otimes \emptyset(y) + \emptyset'(t) \ddot{T} \emptyset(x) \otimes \left[D_{R_2}^{(v_2)} \emptyset(y)\right]\right) + W_b c_b \emptyset'(t)\, \ddot{T} \emptyset(x) \otimes \emptyset(y) = G(x,y,t), \quad \dots (46)$$

where, $G(x,y,t) = Q_{ext}(x,y,t) + Q_{met} + W_b c_b T_a I$.

By collocating Eq. (46) at $M \times (N_1 - 1) \times (N_2 - 1)$ point, as

$$\rho c \emptyset'(t_{\mathcal{T},M,k}) \left[D_{\mathcal{T}}^{(1)}\right]' \ddot{T} \emptyset(x_{R_1,N_1,i}) \otimes \emptyset(y_{R_2,N_2,j}) - K\left(\emptyset'(t_{\mathcal{T},M,k}) \ddot{T} \left[D_{R_1}^{(v_1)} \emptyset(x_{R_1,N_1,i})\right] \otimes \emptyset(y_{R_2,N_2,j}) + \emptyset'(t_{\mathcal{T},M,k}) \ddot{T} \emptyset(x_{R_1,N_1,i}) \otimes \left[D_{R_2}^{(v_2)} \emptyset(y_{R_2,N_2,j})\right]\right) + W_b c_b \emptyset'(t_{\mathcal{T},M,k})\, \ddot{T} \emptyset(x_{R_1,N_1,i}) \otimes \emptyset(y_{R_2,N_2,j}) = G\left(x_{R_1,N_1,i}, y_{R_2,N_2,j}, t_{\mathcal{T},M,k}\right), \quad \dots (47)$$

for $i = 1(1)(N_1 - 1)$, $j = 1(1)(N_2 - 1)$ and $k = 1(1)M$,

where $x_{R_1,N_1,i}$ $(0 \le i \le N_1)$ and $y_{R_2,N_2,j}$ $(0 \le j \le N_2)$ are the shifted Jacobi-Gauss-Lobatto quadrature of $P_{R_1,i}^{(\alpha,\beta)}(x)$ and $P_{R_2,j}^{(\alpha,\beta)}(y)$ respectively, while $t_{\mathcal{T},M,k}$ $(0 \le k \le M)$ are the roots of $P_{\mathcal{T},k}^{(\alpha,\beta)}(t)$, that generates a system of $M \times (N_1 - 1) \times (N_2 - 1)$ nonlinear algebraic equations in the unknown extension coefficients, $\tilde{t}_{kij}$, $i = 1(1)(N_1 - 1)$, $j = 1(1)(N_2 - 1)$ and $k = 1(1)M$, and the rest of this system is obtained from the initial, boundary conditions by utilize Eqs. (2-6), as

$$\left.\begin{array}{l}\emptyset'(0)\,\ddot{T}\emptyset\left(x_{R_1,N_1,i}\right)\otimes\emptyset\left(y_{R_2,N_2,j}\right)= \\ f_0(x_{R_1,N_1,i},y_{R_2,N_2,j}),0\le i\le N_1,0\le j\le N_2 \\ \emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\left[D_{R_1}^{(1)}\emptyset(0)\right]\otimes\emptyset\left(y_{R_2,N_2,j}\right)= \\ g_{10}(y_{R_2,N_2,j},t_{\mathcal{T},M,k}),0\le j\le N_2,0\le k\le M \\ \emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\left[D_{R_1}^{(1)}\emptyset(R_1)\right]\otimes\emptyset\left(y_{R_2,N_2,j}\right)= \\ g_{1R_1}(y_{R_2,N_2,j},t_{\mathcal{T},M,k}),0\le j\le N_2,0\le k\le M \\ \emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\emptyset\left(x_{R_1,N_1,i}\right)\otimes\left[D_{R_2}^{(1)}\emptyset(0)\right]= \\ g_{20}(x_{R_1,N_1,i},t_{\mathcal{T},M,k}),0\le i\le N_1,0\le k\le M \\ \emptyset'\left(t_{\mathcal{T},M,k}\right)\ddot{T}\emptyset\left(x_{R_1,N_1,i}\right)\otimes\left[D_{R_2}^{(1)}\emptyset(R_2)\right]= \\ g_{2R_2}(x_{R_1,N_1,i},t_{\mathcal{T},M,k}),0\le i\le N_1,0\le k\le M\end{array}\right\} \quad \dots (48)$$

This generates $(M+1)\times(N_1+1)\times(N_2+1)$ nonlinear algebraic equations, which can be solved by using a Levenberg-Marquardt MATLAB code algorithm effective and be more robust than other methods (because this algorithm combines the advantages of gradient-descent and Gauss-Newton methods) (21, 22), taking $\ddot{T}$ as its variable, with an initial guess of all zeros, to reduce Eqs. (47-48), consequently, the approximate solution $T_{M,N_1,N_2}(x,y,t)$ at the point $\left(x_{R_1,N_1,i},y_{R_2,N_2,j},t_{\mathcal{T},M,k}\right)$ given in Eq. (38) can be calculated.

Error Bound

Now, an analytic expression will present for the error norm of the preferable approximation for a smooth temperature function $T(x,y,t)\in\Omega$, where $\Omega\equiv[0,R_1]\times[0,R_2]\times[0,\mathcal{T}]$ by its expansion employing triple Jacobi polynomials. This shows an upper bound on the error expected in the numerical solutions. Let at first examine the space

$$\Pi_{M,N_1,N_2}^{\alpha,\beta}=span\left\{P_{R_1,i}^{(\alpha,\beta)}(x)P_{R_2,j}^{(\alpha,\beta)}(y)P_{\mathcal{T},k}^{(\alpha,\beta)}(t)\right\}, \quad i=0(1)N_1, j=0,(1)N_2, k=0(1)M,$$

assume that $T_{M,N_1,N_2}(x,y,t)$ belong to $\Pi_{M,N_1,N_2}^{\alpha,\beta}$, be the preferable approximation of temperature function $T(x,y,t)$. Then depending on the qualifier of the best approximation, have $\forall\theta_{M,N_1,N_2}(x,y,t)\in\Pi_{M,N_1,N_2}^{\alpha,\beta}$

$$\left\|T(x,y,t)-T_{M,N_1,N_2}(x,y,t)\right\|_\infty\le\left\|T(x,y,t)-\theta_{M,N_1,N_2}(x,y,t)\right\|_\infty, \quad \dots (49)$$

appear that the previous inequality be correct if $\theta_{M,N_1,N_2}(x,y,t)$ denotes the interpolating polynomial for $T(x,y,t)$ at points $\left(x_{N_1,i},y_{N_2,j},t_{M,k}\right)$, where $x_{N_1,i},(0\le i\le N_1)$ are the roots of $P_{R_1,N_1+1}^{(\alpha,\beta)}(x)$, $y_{N_2,j},(0\le j\le N_2)$ are the roots of $P_{R_2,N_2+1}^{(\alpha,\beta)}(y)$ and $t_{M,k},(0\le k\le M)$ are the roots of $P_{\mathcal{T},M+1}^{(\alpha,\beta)}(t)$. Then by similar procedures as in (2):

$$\begin{aligned}&T(x,y,t)-\theta_{M,N_1,N_2}(x,y,t)\\&=\frac{\partial^{N_1+1}T(\eta,y,t)}{\partial x^{N_1+1}(N_1+1)!}\prod_{i=0}^{N_1}\left(x-x_{N_1,i}\right)\\&+\frac{\partial^{N_2+1}T(x,\xi,t)}{\partial y^{N_2+1}(N_2+1)!}\prod_{j=0}^{N_2}\left(y-y_{N_2,j}\right)\\&+\frac{\partial^{M+1}T(x,y,\mu)}{\partial t^{M+1}(M+1)!}\prod_{k=0}^{M}\left(t-t_{M,k}\right)\\&-\frac{\partial^{N_1+N_2+M+3}T(\tilde{\eta},\tilde{\xi},\tilde{\mu})}{\partial x^{N_1+1}\partial y^{N_2+1}\partial t^{M+1}}\\&\times\frac{\prod_{i=0}^{N_1}\left(x-x_{N_1,i}\right)\prod_{j=0}^{N_2}\left(y-y_{N_2,j}\right)\prod_{k=0}^{M}\left(t-t_{M,k}\right)}{(N_1+1)!\,(N_2+1)!\,(M+1)!}, \quad \dots (50)\end{aligned}$$

where $\eta,\tilde{\eta}\in[0,R_1]$, $\xi,\tilde{\xi}\in[0,R_2]$ and $\mu,\tilde{\mu}\in[0,\mathcal{T}]$, and can obtain:

$$\begin{aligned}&\left\|T(x,y,t)-\theta_{M,N_1,N_2}(x,y,t)\right\|_\infty\\&\le\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+1}T(\eta,y,t)}{\partial x^{N_1+1}}\right|\frac{\left\|\prod_{i=0}^{N_1}\left(x-x_{N_1,i}\right)\right\|_\infty}{(N_1+1)!}\\&+\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_2+1}T(x,\xi,t)}{\partial y^{N_2+1}}\right|\frac{\left\|\prod_{j=0}^{N_2}\left(y-y_{N_2,j}\right)\right\|_\infty}{(N_2+1)!}\\&+\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{M+1}T(x,y,\mu)}{\partial t^{M+1}}\right|\frac{\left\|\prod_{k=0}^{M}\left(t-t_{M,k}\right)\right\|_\infty}{(M+1)!}\\&+\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+N_2+M+3}T(\tilde{\eta},\tilde{\xi},\tilde{\mu})}{\partial x^{N_1+1}\partial y^{N_2+1}\partial t^{M+1}}\right|\times\\&\frac{\left\|\prod_{i=0}^{N_1}\left(x-x_{N_1,i}\right)\right\|_\infty\left\|\prod_{j=0}^{N_2}\left(y-y_{N_2,j}\right)\right\|_\infty\left\|\prod_{k=0}^{M}\left(t-t_{M,k}\right)\right\|_\infty}{(N_1+1)!\,(N_2+1)!\,(M+1)!} \quad \dots (51)\end{aligned}$$

since $T(x,y,t)$ is a smooth temperature function on Ω, then there exist a constants C_1,C_2,C_3 and C_4, such that:

$$\left.\begin{array}{l}\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+1}T(\eta,y,t)}{\partial x^{N_1+1}}\right|\le C_1\\ \max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_2+1}T(x,\xi,t)}{\partial y^{N_2+1}}\right|\le C_2\\ \max_{(x,y,t)\in\Omega}\left|\frac{\partial^{M+1}T(x,y,\mu)}{\partial t^{M+1}}\right|\le C_3\\ \max_{(x,y,t)\in\Omega}\left|\frac{\partial^{N_1+N_2+M+3}T(\tilde{\eta},\tilde{\xi},\tilde{\mu})}{\partial x^{N_1+1}\partial y^{N_2+1}\partial t^{M+1}}\right|\le C_4\end{array}\right\}. \quad \dots (52)$$

The factor $\left\|\prod_{i=0}^{N_1}\left(x-x_{N_1,i}\right)\right\|_\infty$ minimized as follows: Let we utilize the one-to-one mapping $x=\frac{R_1}{2}(z+1)$ between the intervals $[-1,1]$ and $[0,R_1]$ to deduce that

$$\min_{x_{N_1,i}\in[0,R_1]} \max_{x\in[0,R_1]} \prod_{i=0}^{N_1}(x-x_{N_1,i})$$
$$= \min_{z_{N_1,i}\in[-1,1]} \max_{z\in[-1,1]} \left|\prod_{i=0}^{N_1}\frac{R_1}{2}(z-z_{N_1,i})\right|$$
$$= \left(\frac{R_1}{2}\right)^{N_1+1} \min_{z_{N_1,i}\in[-1,1]} \max_{z\in[-1,1]} \left|\prod_{i=0}^{N_1}(z-z_{N_1,i})\right| \quad \ldots(53)$$
$$= \left(\frac{R_1}{2}\right)^{N_1+1} \min_{z_{N_1,i}\in[-1,1]} \max_{z\in[-1,1]} \left|\frac{P_{N_1+1}^{(\alpha,\beta)}(z)}{\kappa_{N_1}^{(\alpha,\beta)}}\right|,$$

where $\kappa_{N_1}^{(\alpha,\beta)} = \frac{\Gamma(2N_1+\alpha+\beta+1)}{2^{N_1}(N_1)!\,\Gamma(N_1+\alpha+\beta+1)}$ is the leading coefficient of $P_{N_1+1}^{(\alpha,\beta)}(z)$ and $z_{N_1,i}$ are the roots of $P_{N_1+1}^{(\alpha,\beta)}(z)$. It is a well-famed reality (23), that the Jacobi polynomials satisfy

$$\max_{z\in[-1,1]} \left|P_{N_1+1}^{(\alpha,\beta)}(z)\right| \le C_5(N_1+1)^q, \qquad \alpha,\beta > -1,$$

where $q = max(\alpha,\beta,-\frac{1}{2})$ and C_5 is a favorable constant, and reach the maximum of their absolute value on the interval $[-1,1]$, at $z=-1$ provided that $\alpha \ge \beta$ and $\alpha \ge -\frac{1}{2}$ from (24),

$$\max_{z\in[-1,1]} \left|P_{N_1+1}^{(\alpha,\beta)}(z)\right| = P_{N_1+1}^{(\alpha,\beta)}(1) = \frac{\Gamma(N_1+\alpha+2)}{(N_1+1)!\;\Gamma(\alpha+1)} = \mathcal{O}((N_1+1)^{\alpha}),$$

from Eqs. (51-52), get

$$\|T(x,y,t)-T_{M,N_1,N_2}(x,y,t)\|_\infty \le \widetilde{C_1}\frac{\left(\frac{R_1}{2}\right)^{N_1+1}(N_1+1)^q}{\kappa_{N_1}^{(\alpha,\beta)}(N_1+1)!} + \widetilde{C_2}\frac{\left(\frac{R_2}{2}\right)^{N_2+1}(N_2+1)^q}{\kappa_{N_2}^{(\alpha,\beta)}(N_2+1)!} + \widetilde{C_3}\frac{\left(\frac{T}{2}\right)^{M+1}(M+1)^q}{\kappa_{M}^{(\alpha,\beta)}(M+1)!} + \widetilde{C_4}\frac{\left(\frac{R_1}{2}\right)^{N_1+1}\left(\frac{R_2}{2}\right)^{N_2+1}\left(\frac{T}{2}\right)^{M+1}(N_1+1)^q}{\kappa_{N_1}^{(\alpha,\beta)}\kappa_{N_2}^{(\alpha,\beta)}\kappa_{M}^{(\alpha,\beta)}(N_1+1)!(N_2+1)!(M+1)!}. \quad \ldots(54)$$

Hence, an upper bound of the maximum absolute errors achieved for the approximate solution. The convergence of the recommended method depends fundamentally on the above error bound. Moreover, the speed of convergence of “Jacobi collocation methods” was proved be fast for any choice of shifted Jacobi parameters (25, 26).

Estimation of the Error Function

In this section, an efficient error estimation will present for the SJ-GL-Ps and also a technique for obtaining the corrected solution of the M-SFBHE as in equation (1) under the Eqs. (2-6) by using the residual correction method and thus the approximate solution Eq. (38) which corrected by the proposed method (27).

For our aim, let's define $e_{M,N_1,N_2}(x,y,t) = T(x,y,t) - T_{M,N_1,N_2}(x,y,t)$ as the error function of the Collocation approximation $T_{M,N_1,N_2}(x,y,t)$ to $T(x,y,t)$, where $T(x,y,t)$ is the exact solution for the Eq. (1) under Eqs. (2-6). Hence, $T_{M,N_1,N_2}(x,y,t)$ satisfies the following system:

$$L[T_{M,N_1,N_2}(x,y,t)] = \rho c\frac{\partial T_{M,N_1,N_2}(x,y,t)}{\partial t} - K\left(\frac{\partial^{v_1}T_{M,N_1,N_2}(x,y,t)}{\partial x^{v_1}} + \frac{\partial^{v_2}T_{M,N_1,N_2}(x,y,t)}{\partial y^{v_2}}\right) + W_b c_b T_{M,N_1,N_2}(x,y,t) = Q_{ext}(x,y,t) + Q_{met} + W_b c_b T_a + R_{M,N_1,N_2}, \quad \ldots(55)$$

with the initial and boundary conditions

$$T_{M,N_1,N_2}(x,y,0) = T_c, 0 < x < R_1, \quad 0 < y < R_2, \quad \ldots(56)$$
$$-K\frac{\partial T_{M,N_1,N_2}(0,y,t)}{\partial x} = q_0, 0 \le y \le R_2, \quad t > 0, \quad \ldots(57)$$
$$-K\frac{\partial T_{M,N_1,N_2}(x,0,t)}{\partial y} = q_1, 0 \le x \le R_1, \quad t > 0, \quad \ldots(58)$$
$$-K\frac{\partial T_{M,N_1,N_2}(R_1,y,t)}{\partial x} = 0, 0 \le y \le R_2, \quad t > 0, \quad \ldots(59)$$
$$-K\frac{\partial T_{M,N_1,N_2}(x,R_2,t)}{\partial y} = 0, 0 \le x \le R_1, \quad t > 0. \quad \ldots(60)$$

Here, $R_{M,N_1,N_2}(x,y,t)$ is the residual function of the M-SFBHE as in Eq. (1) which obtained by substituting the approximate solution $T_{M,N_1,N_2}(x,y,t)$ into Eqs. (1-6).

Now, let us subtract Eqs. (55-60) from Eqs. (1-6) respectively, Then, the Error equation which obtained is:

$$\rho c\frac{\partial e_{M,N_1,N_2}(x,y,t)}{\partial t} - K\left(\frac{\partial^{v_1}e_{M,N_1,N_2}(x,y,t)}{\partial x^{v_1}} + \frac{\partial^{v_2}e_{M,N_1,N_2}(x,y,t)}{\partial y^{v_2}}\right) + W_b c_b e_{M,N_1,N_2}(x,y,t) = -R_{M,N_1,N_2}, \quad \ldots(61)$$

with the homogeneous conditions:

$$e_{M,N_1,N_2}(x,y,0) = 0, 0 < x < R_1, \quad 0 < y < R_2, \quad \ldots(62)$$
$$-K\frac{\partial e_{M,N_1,N_2}(0,y,t)}{\partial x} = 0, 0 \le y \le R_2, \quad t > 0, \quad \ldots(63)$$
$$-K\frac{\partial e_{M,N_1,N_2}(x,0,t)}{\partial y} = 0, 0 \le x \le R_1, \quad t > 0, \quad \ldots(64)$$

$$-K\frac{\partial e_{M,N_1,N_2}(R_1,y,t)}{\partial x}=0, 0\le y\le R_2, \quad t>0, \qquad \dots(65)$$

$$-K\frac{\partial e_{M,N_1,N_2}(x,R_2,t)}{\partial y}=0, 0\le x\le R_1, \quad t>0. \qquad \dots(66)$$

Finally, the error in Eq. (61-66) will be solved in the same method for solution suggestion and thus the resulting approximation: $e_{M,N_1,N_2}(x,y,t)$ to e_{M,N_1,N_2} as following:

$$e_{M,N_1,N_2}(x,y,t) \approx \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{M}\tilde{\tilde{t}}_{kij}P_{R_1,i}^{(\alpha,\beta)}(x)P_{R_2,j}^{(\alpha,\beta)}(y)P_{\mathcal{T},k}^{(\alpha,\beta)}(t) = \emptyset(t)'\,\tilde{\tilde{T}}\,\emptyset(x)\otimes\emptyset(y). \qquad \dots(67)$$

Of our note to the Eq. (1), while the theoretical solution is not known, thus the maximum absolute error can be estimated approximately by using

$$E_{M,N_1,N_2}(x,y,t) = max\{e_{M,N_1,N_2}(x,y,t), 0\le t\le \mathcal{T}, 0\le x\le R_1, 0\le y\le R_2\}. \qquad \dots(68)$$

The above error estimation depends on the convergence rates of expansion in Jacobi polynomial (22). Therefore, it provided reasonable convergence rates in spatial and temporal discretization.

Numerical Examples

In this section, the approach presented in section (method for solution), has been applied for solving the two dimensional M-SFBHE in the two examples based on SJ-GL-Ps. The two dimensional M-SFBHE were transformed into non-linear algebraic Eqs. (47-48) respectively. The Levenberg-Marquardt MATLAB code technique, taking $\ddot{T}$ as its variable, was used to minimize these equations as a set of least squares problems. This $\ddot{T}$ is then used in Eq. (38) to acquire our approximate surface of $T(x,y,t)$.

By taking in these examples, that $R_1 = R_2 = \mathcal{T} = 1$, $\alpha=\beta=0$ for various choices of v_1, v_2 and use Gauss-Labatto points.

Example1:

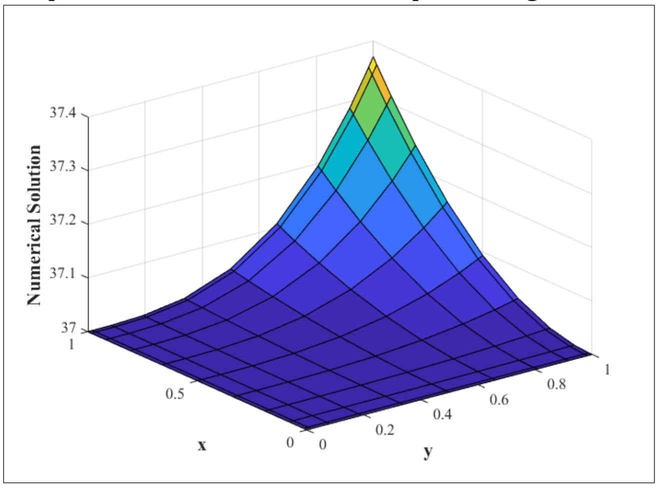
Consider the two dimensional M-SFBHE Eq. (1) case where by choosing Q_{ext} so the exact solution under initial and Neumann boundary conditions is:

$$T(x,y,t) = e^{-t}x^2y^3 + 37. \qquad \dots(69)$$

Table2. Maximum errors obtained for Example 1 with $v_1 = 1.9999$ and $v_2 = 1.6$.

$N_1 = N_2 = M$	Maximum Error
2	3.687987900278245e-04
3	3.638748477925446e-04
4	2.600028022214929e-05
5	2.043736270707086e-05
6	2.694529089808384e-05
7	2.938213830816494e-05
8	3.094610488574290e-05
9	3.204782999688405e-05
10	3.285274994624388e-05

Table 2 show that the maximum errors satisfy from solving the problem under SJ-G-LPs study on $x\in[0,R_1], y\in[0,R_2]$ and $t\in[0,\mathcal{T}]$ when $N_1=N_2=M$ = 2,3,4,5,6,7,8,9 and 10. Figure 1 clarify a comparison between then numerical and exact solutions of Example 1. Figure 2 indicate the maximum error values are observed to be of a low error for all sample sizes, with the best performance occurring for $N_1=N_2=M=5$ ($N_1=N_2=M=2$) at just under 2.1×10^{-5} (3.7×10^{-4}) respectively.



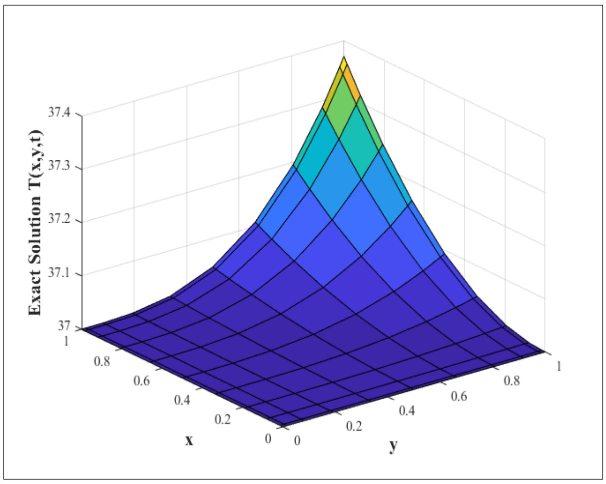


Figure 1. Numerical and exact solutions for Example 1 at $v_1 = 1.9999, v_2 = 1.6$ $\mathcal{T} = R_1 = R_2 = 1,\ N_1 = N_2 = M = 10$.

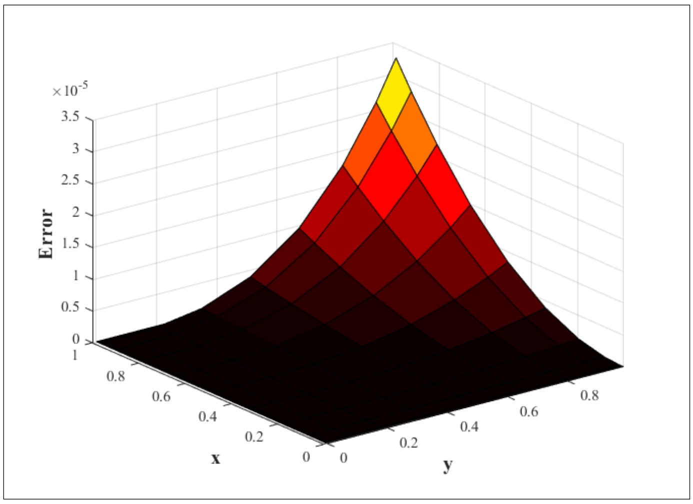


Figure 2: Maximum error for Example 1 at $v_1 = 1.9999, v_2 = 1.6$ $\mathcal{T} = R_1 = R_2 = 1, N_1 = N_2 = M = 10.$

Example 2:

Consider the two dimensional M-SFBHE Eq. (1) case where by choosing Q_{ext} so the exact solution under initial and Neumann boundary conditions **is:**

$$T(x,y,t) = t^4(1 - x^2y^3) + 37. \qquad \dots (70)$$

Table 3. Maximum errors obtained for Example 2 with $v_1 = 1.6$ and $v_2 = 1.9$

$N_1 = N_2 = M$	Maximum Error
2	2.249997568501527e-01
3	4.875079414209438e-02
4	3.660574066088884e-10
5	4.734701519737428e-10
6	5.071569830761291e-10
7	5.086349119665101e-10
8	4.956604016115307e-10
9	4.692637389780430e-10
10	4.219486982037779e-10

Table 3 show that the maximum errors satisfy from solving the problem under SJ-G-LPs study on $x \in [0, R_1], y \in [0, R_2]$ and $t \in [0, \mathcal{T}]$ when $N_1 = N_2 = M = 2,3,4,5,6,7,8,9$ and 10. Figure 3 clarify a comparison between then numerical and exact solutions of Example 2. Figure 4 indicate the maximum error values are observed to be of a low error for all sample sizes, with the best performance occurring for $N_1 = N_2 = M = 4$ ($N_1 = N_2 = M = 2$) at just under 3.7×10^{-10} (2.5×10^{-1}) respectively.

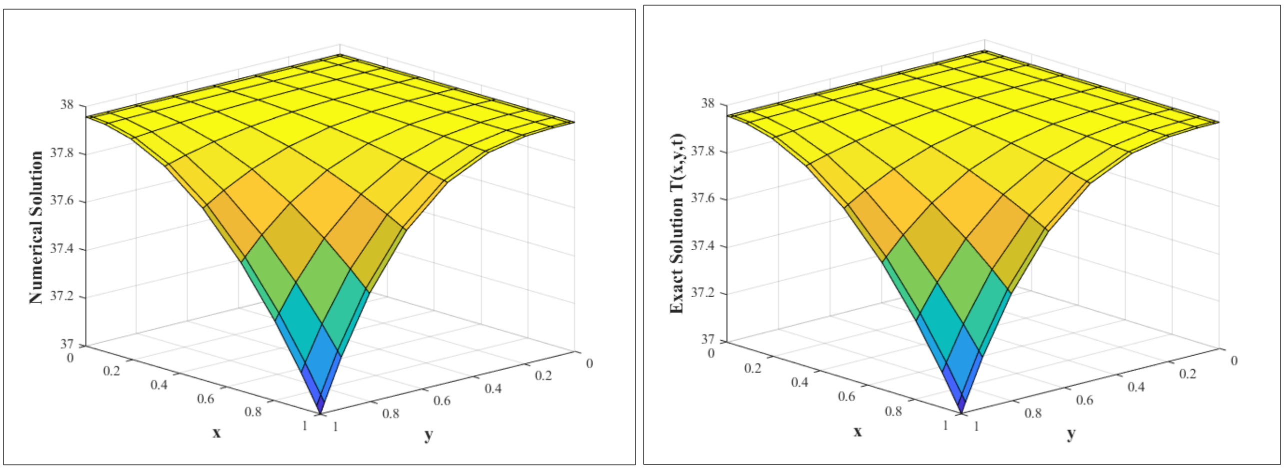


Figure 3. Numerical and exact solutions for Example 2 at $v_1 = 1.6,\ v_2 = 1.9$ $\mathcal{T} = R_1 = R_2 = 1,\ N_1 = N_2 = M = 10$.

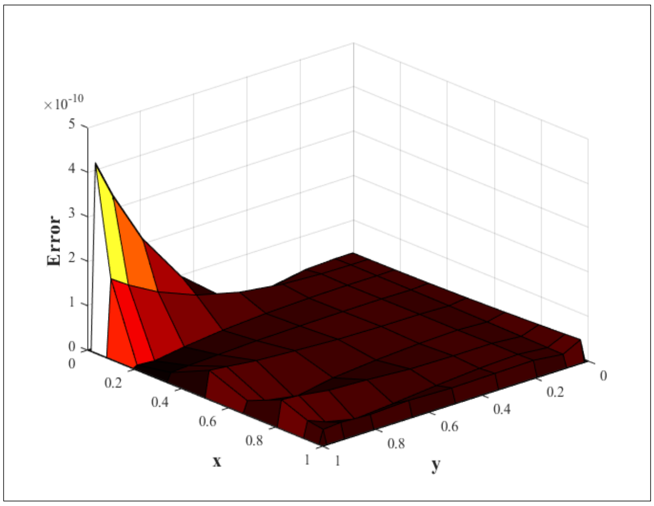


Figure 4: Maximum error for Example 2 at $\upsilon_1 = 1.6, \upsilon_2 = 1.9$ $\mathcal{T} = R_1 = R_2 = 1, N_1 = N_2 = M = 10$.

Conclusions

In this article, an approximate approach for solving two-dimensional M-SFBHE has been introduced. The fractional derivatives are described in the Caputo form. The proposed technique depends on the collocation method of operational matrix formula for the shifted Jacobi-Gauss-Lobatto polynomials. The error of the approximate solution is estimated theoretically and the convergence average of the suggested approach in both spatial and temporal nodes graphically is investigated analyzed. The approximate calculations show that the present technique has higher accurate, good convergence (depending on Figures 2 and 4) by using few grid points.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Basrah.

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التقنية العددية المعتمدة على متعددات حدود جاكوبي- كاوس- لوباتو لحل معادلة الحرارة الحيوية ثنائية البعد متعددة الرتبة الكسورية المكانية

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الخلاصة:
يتناول هذا البحث، الخوارزمية التقريبية لحل معادلة الحرارة الحيوية ثنائية البعد متعددة الرتبة الكسورية المكانية (M-SFBHE). سوف نوسع تطبيق طريقة التجميع لتقديم التقنية العددية لحل M-SFBHE مؤسسة على متعددات حدود جاكوبي- كاوس- لوباتو (SJ-GL-Ps) بالصيغة المصفوفية. استخدمنا صيغة Caputo لتقريب المشتقة الكسرية و لإثبات فائدتها ودقتها، طبقنا الخوارزمية المقترحة على مثالين. النتائج العددية أظهرت أن النهج المستخدم فعال للغاية ويعطي دقة عالية وتقارب جيد.

الكلمات المفتاحية: الدقة، طريقة التجميع، Shifted Jacobi-Gauss-Lobatto polynomials، معادلة الحرارة الحيوية الكسورية.