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Some Results on Normalized Duality Mappings and Approximating Fixed Points in Convex Real Modular Spaces

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Abstract:

In this paper, the concept of normalized duality mapping has introduced in real convex modular spaces. Then, some of its properties have shown which allow dealing with results related to the concept of uniformly smooth convex real modular spaces. For multivalued mappings defined on these spaces, the convergence of a two-step type iterative sequence to a fixed point is proved.

Keywords: Fixed Points; Iterative Sequences; Multivalued Mappings; Real Modular Spaces; Uniformly Convex.
MSC: 49J40; 47J20

Introduction

Nakano in 1950 (1) introduced the concept of a modular on a linear space and refined by Musielak and Orlicz in 1959 (2):

Definition 1 Let M be real linear space. A functionγ:$M \to (0,\infty)$is called modular if

(i) $\gamma\ (v) = 0$if and only if$v = \ 0, v \in M,$;

(ii) $\gamma(\alpha v)$ = $\alpha(v)$ for $\alpha \in F$ with $|\alpha|$ =1, for all $v \in M, \alpha \in R$;

(iii) $\gamma(\alpha v +\ \beta u) \leq \gamma(v)$+ $\gamma(u)$ iff $\alpha + \beta = 1, \alpha,\beta \geq 0$, for all $u,v \in M$.

If (iii) replaced by

(iii)\` $\gamma(\alpha v + \beta u) \leq\ \alpha\gamma(v)$+$\beta\gamma(u)$, for $\alpha,\beta \geq 0$, $\alpha + \beta = 1$, for all $u, v \in M$, then γ is called convex modula

Definition 2 (1) A corresponding modular space, M_γ, defined by γ is

$M_\gamma = \{v \in M\colon \gamma(\alpha v) \to 0$ whenever $\alpha \to 0\}$.

Many works can be found in (3-6).

Definition 3 (7) The γ-ball, $B_r(u)$centered at $u\ \in M_\gamma$with radius $r > 0$as

$B_r(u) = \{\boldsymbol{v}\ \in M_\gamma; \gamma(u - v) < r\}$.

The collection of all γ-balls in a modular space M_γ generates a locally convex Hausdorff topological linear space (7).

Definition 4 (8)

(1) A sequence $\{v_n\} \subset M_\gamma$ is said to be

(a) γ-convergent to $v \in M_\gamma$ and write $v_n \xrightarrow{\gamma} v$ if $\gamma(v_n - v) \to 0$ as n→ ∞.

(b) $\gamma-$ Cauchy whenever $\gamma(v_n - v_m) \to 0$ as $n\ ,m \to \infty$.

(2) If any $\gamma-$ Cauchy sequence in M_γ is $\gamma-$convergent then M_γ is called $\gamma-$ complete.

(3) If for any sequence $\{v_n\} \subset B \subset M_\gamma$ is $\gamma-$convergent to a point in B then B is called $\gamma-$closed.

(3) if any sequence $\{\ v_n\ \} \subset B\ \subset M_\gamma$ has a $\gamma-$convergent subsequence then $B$is called $\gamma-$compact.

(4) if $daim_\gamma(B) = sup\{\gamma(v - u);\ v,u \in B\} < \infty$ then $B$is called $\gamma-$ bounded, where $daim_\gamma(B) =$
is called the $\gamma-$ diameter of B.

(5) The distance between $v \in M_\gamma$ and $B \subset M_\gamma$ is $\gamma(v - B)$ =inf $\{\gamma(v - u); u \in B\}$.

Definition 5 (8) Let $A\ ,B$ be two non – empty subsets in 2^{M_γ} then H_γ (A,B) = $max\{ sup_{a\in A}\gamma(a - B\ ),\ sup_{b\in B}\ \gamma(b - A)\}$ is Hausdorff distance of A and B.

Directly, the following lemma is obtained

Lemma 1 Let M_γ be a modular space and $(A_n)$and (B_n) real sequences in $CB(M_\gamma)$.Then there is a sequence (a_n) in A_n ,(b_n) in B_n such that

$\gamma\,(a_n - b_n) = H_\gamma\,(A_n, B_n) + \varepsilon_n$ $, lim_{n\to\infty} \varepsilon_n = 0$ ... (1)

Definition 6 (9) A point $x \in A$ is a fixed point of a multivalued $T: A \longrightarrow 2^A \Leftrightarrow x \in T(x)$. If T is single-valued mapping, x is a fixed point $\Leftrightarrow x = T(x)$.

Definition 7 (8) M_γ is called uniformly convex if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, $\ni$ if $\gamma(v) = \gamma(u) = 1$ and $\gamma(v-u) \geq \varepsilon$, then $\gamma\left(\frac{1}{2}(v+u)\right) \leq 1-\delta$.

Definition 8 (8) μ_γ is said to be strictly convex if $u, v \in S(u)$ with $\gamma(u) = \gamma(v) = \gamma((u+v)/2)$, Then $u = v$

Example 1: Let $\mu_\gamma = R^n$ with convex modular $\gamma(u) = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}$, $u = (u_1, \dots, u_n) \in R^n$. Then μ_γ is strictly convex.

Example 2: Let $\mu_\gamma = R^n$ with convex modular $\gamma(u) = |u_1| + \cdots + |u_n|$, $u = (u_1, \dots, u_n) \in R^n$ Then μ_γ is not strictly convex, since $u = (1,0,0,\dots,0)$ and $v = (0,1,0,\dots,0)$, implies that $u \neq v$, $\gamma(u) = 1 = \gamma(v)$ but $\gamma(u+v) = 2$

Example 3: Let $\mu_\gamma = R^n$ with convex modular $\gamma(u) = \max_{1\leq i\leq n}|u_i|$ $u = (u_1, \dots, u_n) \in R^n$. Then μ_γ is not strictly convex, let $u = (1,0,0,\dots,0)$ and $v = (1,1,0,\dots,0)$ and $\neq v$, $\gamma(u) = 1 = \gamma(v)$, but $\gamma(u+v) = 2$.

Now, consider $\mu_\gamma^* = \{h; h: \mu_\gamma \to R$ is bounded and linear$\}$ and $\gamma^*: \mu_\gamma^* \to [-\infty, \infty]$, $\gamma^*(h) = \sup\{|h(u)|: u \in \mu_\gamma, \gamma(u) = 1\}$

Proposition1 (8): (μ_γ^*, γ^*) is complete convex real modular, μ_γ^* is the dual space of μ_γ.

Recall the Hahn – Banach Theorem for linear spaces

Theorem 1 (10)**:** Let C be a subspace of linear space μ, g a sublinear functional on μ and f a linear functional on $C \ni f(u) \leq g(u), \forall u \in C$ Then $\exists F \ni F(u) \leq g(u), u \in \mu$**.**

Corollary 1: Let C be a subspace of real modular space μ_γ and f a bounded linear functional on C. Then there exists a bounded linear functional F defined on μ_γ that is an extension of $f \ni \gamma^*(F) = \gamma|_C(f)$.

Proof: By Theorem1,$\exists$, F is a linear functional on μ_γ such that $F(u) = f(u)$, $\forall u \in \mu_\gamma$. $F(u) \leq g(u) = \gamma^*(f)\gamma(u)$, $\forall u \in \mu_\gamma$. So, $\gamma^*(F) = \sup\{|F(u)| : \gamma(u) = 1\} \leq \sup\{|g(u)| : \gamma(u) = 1\}$. $\gamma^*(F) = \sup\{|F(u)|: \gamma(u) = 1\} \leq \sup\{\gamma^*(f)\gamma(u): \gamma(u) = 1\}$. $\gamma^*(F) = \sup\{|F(u)|: \gamma(u) = 1\} \leq \sup\{\gamma^*(f): \gamma(u) = 1\}$, thus, $\gamma^*(F) \leq \gamma^*(f)$. But, $\gamma^*(F) \geq \gamma^*(f) \Rightarrow \gamma^*(F) = \gamma^*(f)$.

As a consequence

Corollary 2: Let u be an element of a modular space μ_γ. Then $\exists h \in \mu_\gamma^*$ $(h \neq 0) \ni h(u) = \gamma^*(h)\,\gamma(u)$ and $\gamma^*(h) = \gamma(u)$.

Corollary 3: Let $0 \neq u \subset \mu_\gamma$ then $\exists g \in \mu_\gamma^*$ $\ni g(u) = \gamma(u)$ and $\gamma^*(g) = 1$.

Proof: By Corollary 2, g has a linear extension $g^\sim$ such that $\gamma^*(g^\sim) = \gamma^*(g) = 1$. So, $g^\sim(u) = g(u) = \gamma(u)$.

For more background in normed spaces, see (9-11). This paper is devoted to presenting several cases of the real modular spaces related to convexity and smooth convexity to be used in the study of the convergence for two types of iterative sequences. The concepts of γ -normalized duality and uniformly smooth convex mapping in the real convex modular spaces are presented with some interesting properties. Some important results concerning the $\gamma-$ convergence of two-step iteration for multivalued mappings are established dealing with Lipschitzian strongly pseudo-contractions and strongly pseudo-contractions.

Normalized duality mappings

This section will include introducing many new definitions and facts in modular spaces. Starting with the concept of duality pairing.

Definition 8: A modular duality pairing, shortly, (γ - duality pairing) is defined as $\langle .|.\rangle: \mu_\gamma \times \mu_\gamma^* \to R \ni \langle u|h\rangle = h(u)$, for all $u \in \mu_\gamma$ and $h \in \mu_\gamma^*$

As a consequence of this definition, the following proposition achieved:

Proposition 2: Let $\langle .|.\rangle$ be a γ – duality pairing on $\mu_\gamma \times \mu_\gamma^*$, then

i- $\langle au + bv|h\rangle = a\langle u|h\rangle + b\langle v|h\rangle$

ii- $\langle u|ah_1 + bh_2\rangle = a\langle u|h_1\rangle + b\langle u|h_2\rangle$

iii- $\langle u|h\rangle = 0$ for all $u \in \mu_\gamma$ $h = 0$

iv- $\langle u|h\rangle = 0$ for all $h \in \mu_\gamma^*$ $u = 0$

Definition 9: Let μ_γ^* be the dual of a modular space μ_γ .Then $h: \mu_\gamma \to 2^{\mu_\gamma^*}$ is said to be a γ – normalized duality mapping if $H(u) = \{h \in \mu_\gamma^* : < u|h > = \gamma^2(u) = \gamma^{*2}(h)\}$

Theorem 2: Let μ_γ be a convex modular space and $H: \mu_\gamma \to 2^{\mu_\gamma^*}$ be γ –normalized duality mapping. Then

(i) $H(0) = \{0\}$

(ii) $H(u)$ is nonempty closed convex and bounded.

Proof: for (i) $\forall u \in \mu_\gamma$, $H(u) \subseteq \mu_\gamma^*$ $H(0) = \{h \in \mu_\gamma^*: \langle u|h\rangle = \gamma(u)^2 = \gamma^*(h)^2\} = \{h \in \mu_\gamma^* : \langle 0|h\rangle = \gamma(0) = \gamma^*(h)\} = \{0\}$. For part (ii), if $u = 0$ holds by part(i). If $u \neq 0$, then by Theorem

1, $\exists f \in \mu_\gamma^*$ such that $\langle u|f\rangle = \gamma(u)$ and $\gamma^*(f) = 1$. Set $h = \gamma(u) f$. Then $\langle u|h\rangle = \gamma(u)\langle u|f\rangle = \gamma(u)^2$ and $\gamma^*(h) = \gamma(u)$, and it follows that $Hu \neq 0$, for each $u \neq 0$. Now, suppose $f_1, f_2 \in H(u)$ and $t \in (0,1)$, because $\langle u|f_1\rangle = \gamma(u)\gamma^*(f_1)$, $\gamma(u) = \gamma^*(f_1)$ and $\langle u|f_2\rangle = \gamma(u)\gamma^*(f_2)$, $\gamma(u) = \gamma^*(f_2)$

$\langle u|tf_1 + (1-t)f_2\rangle \leq \gamma^*(tf_1 + (1-t)f_2)\gamma(u)$

$\leq (t\gamma^*(f_1) + (1-t)\gamma^*(f_2))\gamma(u)$

$= \gamma^2(u)$

Then

$\gamma^2(u) \leq \gamma(u)\gamma^*(tf_1 + (1-t)f_2) \leq \gamma^2(u)$ and $\gamma^2(u) = \gamma(u)\gamma^*(tf_1 + (1-t)f_2)$, i.e., $\gamma(u) = \gamma^*(tf_1 + (1-t)f_2)$. Thus, $\langle u|tf_1 + (1-t)f_2\rangle = \gamma(u)\gamma^*(tf_1 + (1-t)f_2)$ and $\gamma(u) = \gamma^*(tf_1 + (1-t)f_2)$. This means that $tf_1 + (1-t)f_2 \in H(u)$, so, $H(u)$ is a convex set.

Let $h \in H^{`}(u)$ (h is accumulation Point of $H(u)$) , $\exists \{h_n\}$ a sequence in $H(u)$ $\ni h_n \to h$. The definition of $H(u)$ implies that $\langle u|h_n\rangle = \gamma(u)^2 = \gamma^*(h_n)^2$. By taking the limit $\Rightarrow \langle u|h\rangle = \gamma(u) = \gamma(h)$. So, $h \in H(u)$, then, $H(u)$ is closed set. Finally, since $\gamma(u) = M \geq 0$ and $\gamma(h) \leq M$ then $H(u)$ is bounded.

Theorem 3: Let μ_γ H: $\mu_\gamma \to 2^{\mu_\gamma^*}$ be $\gamma-$normalized duality mapping. Then, if μ_γ^* is strictly convex space, H is single-valued.

Proof: Let $h_1, h_2 \in H(u)$ for $u \in \mu_\gamma$. Then $\langle u|h_1\rangle = \gamma^{*2}(h_1) = \gamma^2(u)$ and $\langle u|h_2\rangle = \gamma^{*2}(h_2) = \gamma^2(u)$. Adding the above identities $\Rightarrow \langle u|h_1 + h_2\rangle = 2\gamma^2(u)$. Because of $2\gamma^2(u) = \langle u|h_1 + h_2\rangle \leq \gamma(u)\gamma^*(h_1 + h_2)$.This implies that $\gamma^*(h_1) + \gamma^*(h_2) = 2\gamma(u) \leq \gamma^*(h_1 + h_2)$. It now follows from the fact $\gamma^*(h_1 + h_2) \leq \gamma^*(h_1) + \gamma^*(h_2)$ that $\gamma^*(h_1 + h_2) = \gamma^*(h_1) + \gamma^*(h_2)$. Because μ_γ^* is strictly convex and $\gamma^*(h_1 + h_2) = \gamma^*(h_1) + \gamma^*(h_2)$, then there exists $\lambda \in R$ such that $h_1 = \lambda h_2$. Because $\langle u|h_2\rangle = \langle u|h_1\rangle = \langle u|\lambda h_2\rangle = \lambda\langle u|h_2\rangle$, this implies that $\lambda = 1$ and hence $h_1 = h_2$. Therefore, H is single-valued.

Let ∂C be the boundary C subset of μ_γ, $S\mu_\gamma$ be the unit sphere of μ_γ and $S\mu_\gamma = \{u \in \mu_\gamma : \gamma(u) = 1\}$.

Definition 10: μ_γ is said to be smooth if for each $u \in S\mu_\gamma$, there exists a unique functional $h_x \in \mu_\gamma^*$ $\ni \langle u|h_x\rangle = \gamma(u)$ and $\gamma(h_x) = 1$.

Theorem 4: μ_γ is smooth, if μ_γ^* is strictly convex and it is strictly convex if μ_γ^* is smooth.

Proof: If μ_γ is not smooth, then $\exists u_0 \in S\mu_\gamma$, and two functional $u^* \neq v^*$ in $S_{\mu_\gamma^*}$ with $u^*(u_0) = v^*(u_0) = 1$ but this means that $\gamma(u^* + v^*) \geq (u^* + v^*)(u_0) = 2$ which implies that μ_γ^* is not strictly convex. If μ_γ is not strictly convex then there exists $u \neq v$ in $S\mu_\gamma$ so that $\gamma(\lambda u + (1-\lambda)v) = 1$, for all $0 \leq \lambda \leq 1$. Let $u^* \in S_{\mu_\gamma^*}$ such that $u^*\left(\frac{u+v}{2}\right) = 1$. Which implies that $u^*(u) = u^*(v) = 1$, which by viewing u and v to be elements in μ_γ^{**}, implies that $\mu_\gamma^*$is not smooth.

Definition 11: Let μ_γ be $\varphi: \mu_\gamma \to (-\infty, \infty]$ a function. Then the limit

$$\lim_{t\to 0}\frac{\varphi(u+tv) - \varphi(u)}{t} = inf_{t>0}\frac{\varphi(u+tv) - \varphi(u)}{t}$$

if it exists, is called a γ -directional derivative of φ at $u \in \mu_\gamma$ in the direction $v \in \mu_\gamma$, it is denoted by $\varphi'(u, v)$.

The function φ is called γ - Gateaux differentiable at a point $u \in \mu_\gamma$ if there exists a continuous linear functional h on μ_γ such that $\langle v|h\rangle = \varphi'(u, v)$ for all $\in \mu_\gamma$. The element h denoted by $\varphi'(u) or\ \varphi(u)$. From the definition of γ-Gateaux derivative of φ that

(i) $\varphi'(u)(0) = 0$,

(ii) $\varphi'(u)(\lambda v) = \lambda \lim_{n\to\infty}\frac{\varphi(u+t\lambda v)-\varphi(u)}{t} = \lambda\varphi'(u)(v)\ for\ all\ \lambda \in R$

Remark 1: If the function φ is γ -Gateaux differentiable at $u \in \mu_\gamma$, then there exists $h = \varphi'(u) \in \mu_{\gamma^*}$ such that

$$\frac{d}{dt}\varphi(u+tv)\Big|_{t=0}\langle v|\varphi'(u)\rangle = \langle v|h\rangle \text{ for all } v \in \mu_\gamma .$$

The function φ is said to be γ-Frechet differentiable at a point $u \in \mu_\gamma$ if there exists a continuous linear functional h on μ_γ such that

$$\lim_{\gamma(v)\to 0}\frac{|\varphi(u+v) - \varphi(u) - \langle v|h\rangle|}{\gamma(v)} = 0$$

Definition 12: Let μ_γ be a convex real modular space. Then a function $\rho\mu_\gamma: R^+ \to R^+$is said to be the modulus of smoothness of μ_γ if

$$\rho\mu_\gamma(t) = \sup\{\frac{\gamma(u+v)+\gamma(u-v)}{2} - 1 : \gamma(u) = 1, \gamma(v) = t\}$$
$$= \sup\left\{\frac{\gamma(u+tv)+\gamma(u-tv)}{2} - 1 : \gamma(u) = \gamma(v) = 1\right\}, t \geq 0$$

Theorem 5: Let μ_γ be a convex real modular space. Then:

(a) $\rho\mu_{\gamma^*}(t) = \sup\left\{\frac{t\varepsilon}{2} - \delta\mu_\gamma(\varepsilon): 0 \le \varepsilon \le 2\right\}$ for all $t > 0$.

(b) $\rho\mu_\gamma(t) = \sup\left\{\frac{t\varepsilon}{2} - \delta\mu_{\gamma^*}(\varepsilon): 0 \le \varepsilon \le 2\right\}$ for all $t > 0$.

Proof (a): By the definition of modulus of smoothness of μ_{γ^*}, the following holds
$2\rho\mu_{\gamma^*}(t) = \sup\{\gamma^*(u^* + tv^*) + \gamma^*(u^* - tv^*) - 2 : u^*, v^* \in S\mu_{\gamma^*}\}$
$= \sup\{\langle u|u^*\rangle + t\langle u|v^*\rangle + \langle v|u^*\rangle - t\langle v|v^*\rangle - 2: u, v \in S\mu_\gamma, u^*, v^* \in S\mu_{\gamma^*}\}$
$= \sup\{\gamma(u+v) + t\,\gamma(u-v) - 2: u, v \in S\mu_\gamma\}$
$= \sup\{\gamma(u+v) + t\varepsilon - 2: u, v \in S\mu_\gamma, \gamma(u-v) = \varepsilon,\ 0 \le \varepsilon \le 2\}$
$= \sup\{t\varepsilon - 2\delta\mu_\gamma(\varepsilon):\ 0 \le \varepsilon \le 2\}$.
(***b***) Obtained in the same manner.

Definition 13: A convex real modular space μ_γ is said to be uniformly smooth if $\rho'\mu_\gamma(0) = \lim_{t\to 0}\frac{\rho\mu_\gamma(t)}{t} = 0$.

Theorem 6: Every uniformly smooth convex real modular space μ_γ is smooth.

Proof: Suppose that μ_γ is not smooth. Then there exists $u \in \mu_\gamma\backslash\{0\}$, and $h, g \in \mu_{\gamma^*}$ $\ni h \ne g$, $\gamma(h) = \gamma(g) = 1$ and $\langle u|h\rangle = \langle u|g\rangle = \gamma(u)$. Let $y \in \mu_\gamma$ such that $\gamma(v) = 1$ *and* $\langle v|h - g\rangle > 0$.
For each $t > 0$, $0 < t\langle v|h - g\rangle = t\langle v|h\rangle - t\langle v|g\rangle$
$= \frac{\langle u+tv|h\rangle\langle u-tv|g\rangle}{2} - 1 \le \frac{\gamma(u+tv)+\gamma(u-tv)}{2} - 1,$
And it follows that $0 < \langle v|h - g\rangle \le \frac{\rho\mu_\gamma(t)}{t}$, for each $t > 0$. Hence μ_γ is not uniformly smooth.

Definition 14: Let μ_γ be a convex real modular space. Then the modular γ of μ_γ is γ - Gateaux differentiable at a point $u_0 \in S\mu_\gamma$ if for $v \in S\mu_\gamma$

$$\frac{d}{dt}\left(\gamma(u_0 + tv)\right)|_{t=0} = \lim_{t\to 0}\frac{\gamma(u_0+tv) - \gamma(u_0)}{t}$$

Corollary 4: Let $\varphi: \mu_\gamma \to (-\infty, \infty]$ be a proper convex function. Then φ is γ -G$a$teaux differentiable at $u \in int(dom(\varphi))$ $\Leftrightarrow$it has a unique subgradient $\partial\varphi(u) = \{\varphi'(u)\}$, i.e,
$\frac{d}{dt}\varphi(u + tv)|_{t=0} = \langle v|\partial\varphi(u)\rangle = \langle v|\varphi'(u)\rangle$ for all $v \in \mu_\gamma$.

Theorem 7: μ_γ is smooth $\Leftrightarrow\gamma$ is γ -G$a$teaux differentiable on $\mu_\gamma\backslash\{0\}$.

Proof: Since φ is a proper convex continuous function, γ is γ -G$a$teaux differentiable if and only if it has a unique subgradient, γ is γ -G$a$teaux differentiable at u
$\Leftrightarrow \partial\gamma(u) = \{h \in \mu_\gamma^* : \langle u|h\rangle = \gamma(u), \gamma^*(u) = 1\}$ is singleton
$\Leftrightarrow$ there exists a unique $h \in \mu_\gamma^*$ such that $\langle u|h\rangle = \gamma(u)$ *and* $\gamma^*(h) = 1$
$\Leftrightarrow$ smooth.

Proposition 2: Let (μ_γ, γ) be a smooth space of μ_γ, suppose one of the following holds:
(1) H is uniformly continuous on a bounded subset of μ_γ
(2) $\langle H(u) - H(v)|u - v\rangle = \le \gamma(u - v)^2$, $\forall u, \forall v$ in μ_γ
(3) For any bounded subset D of μ_γ $\ni \langle H(u) - H(v)\,|\,u - v\rangle = \le c(\gamma(u - v))$, $\forall u, \forall v$ in D where c satisfies $\lim_{t\to 0}\frac{c(t)}{t} = 0$
Then, for $\varepsilon > 0$ and a bounded subset C there is $\delta > 0$ such that
$\gamma(tu + (1-t)v)^2 \le 2\langle H(v)|\,u\rangle t + 2\varepsilon t + (1-2t)\gamma(v)^2$, for any $u, v \in C$ and $t \in [0, \delta)$

Proof: Since (2) implies (3), this proposition will be proven under the condition (3).
$\frac{1}{2}\frac{d}{dt}\gamma(tu + (1-t)v)^2 = \langle H(tu + (1-t)v)|\,u - v\rangle, u, v \in \mu_\gamma$ … (2)
Since, $\lim_{t\to 0}\frac{c(t)}{t} = 0$, where is $\delta` > 0$ such that for $t \in [0, \delta)$ and $\frac{c(t)}{t} \le \frac{\varepsilon}{diam} C$.Hence, for any $u, v \in C$,$\langle H(tu + (1-t)v)|\,t(u - v)\rangle - \langle H(v)|\,t(u - v)\rangle =$
$\langle H(tu + (1-t)v) - H(v)|\,t(u - v)\rangle$
$\le c\left(t\gamma(u - v)\right)$.
Therefore,
$\langle H(tu + (1-t)v) - H(v)|\,t(u - v)\rangle \le c\left(t\gamma(u - v)\right)/t$
$= (c(t\gamma(u - v))/t\gamma(u - v))\,\gamma(u - v)$
$\le (c(t\gamma(u - v))/t\gamma(u - v))$ diam C … (3)
Let $\delta = \delta`/\text{diam}\,C$, from (3) $\Rightarrow \langle H(tu + (1-t)v) - H(v)|\,t(u - v)\rangle < \varepsilon$ For any t$\in [0, \delta)$. So, $\langle H(tu + (1-t)v) - H(v)|\,t(u - v)\rangle \le \langle H(v)|\ u - v\rangle + \varepsilon$
From (2) $\Rightarrow$ $\frac{1}{2}\frac{d}{dt}\gamma(tu + (1-t)v)^2 \le \langle H(v)|\ u - v\rangle \le +\varepsilon$.

The following lemmas obtain by definition of γ – duality pairing

Lemma 2: For any $u, v \in \mu_\gamma$, $\langle h(u)|v\rangle \le \gamma(u)\gamma(v)$, $\forall h(u) \in H(u)$.

Lemma 3: For any $u, v \in \mu_\gamma$, $\gamma^2(u + v) \le \gamma^2(u) + 2\langle v|h(u + v)\rangle$, $\forall u, v \in \mu_\gamma$, $\forall h(u + v) \in H(u + v)$.

Proof: $\varphi(u) = \frac{1}{2}\gamma^2(u)$, $H(u) = \partial\varphi(u) = \{ f \in \mu^*_\gamma : \varphi(v) - \varphi(u) \geq (f, v-u), \forall v \in \mu_\gamma \}$. It follows from the definition of subdifferential of φ that $\varphi(u) - \varphi(u+v) \geq \langle h | u - (u+v)\rangle = -\langle h|v\rangle, \forall h \in H(u+v)$.
Hence, $\gamma^2(u+v) \leq \gamma^2(u) + 2\langle v | h(u+v)\rangle, \forall h(u+v) \in H(u+v)$.

Convergence Theorems

Definition 15: Let $\emptyset \neq A \subset \mu_\gamma$. A mapping $T: A \to 2^{\mu_\gamma}$ is strongly accretive if $\exists k > 0 \ni \forall u, \forall v \in A$, $\forall \xi \in Tu$, $\forall \mu \in Tv, \exists h(u-v) \in H(u-v)$ with

$\langle \xi - \mu | h(u-v)\rangle \geq k\gamma^2(u-v)$ ... (4)

(or respectively $\langle Tu - Tv | h(u-v)\rangle \geq k\gamma^2(u-v)$ … (5)

If $k = 0$, Then T is said to be multivalued accretive.

Definition 16: Let $\emptyset \neq A \subset \mu_\gamma$. A mapping $T: A \to 2^{\mu_\gamma}$ is said to be a multivalued strongly pseudo contraction. If $\exists t > 1$, $\ni \forall u, v \in A, \forall \xi \in Tu$, $\mu \in Tv$ there exist $h(u-v) \in H(u-v)$ with

$\langle \xi - \mu | h(u-v)\rangle \leq \frac{1}{t}\gamma^2(u-v)$ … (6)

(or respectively $\langle Tu - Tv | h(u-v)\rangle \leq \frac{1}{t}\gamma^2(u-v)$ … (7)

Definition 17 (12): Let $\emptyset \neq A \subset \mu_\gamma$. A mapping $T: A \to 2^{M_\gamma}$ such that

$H_\gamma(Tu, Tv) \leq K\gamma(u-v)$, for all $u, v \in A$ …(8)

is said to be multivalued Lipschitz if there exists $k > 0$ and multivalued contraction if $k < 1$.

Let $u_0 \in A$ and $T: A \longrightarrow 2^A$. Define a sequence $\{u_n\} \subset A$ by

$u_{n+1} \in (1 - a_n)u_n + a_n T v_n$
$v_n \in (1 - \beta_n)u_n + \beta_n Tu_n$, $\forall n \geq 0$ … (9)

or $u_{n+1} = (1 - a_n)u_n + a_n\mu_n$, $\mu_n \in Tv_n$, $\forall n \geq 0$
$v_n = (1 - \beta_n)u_n + \beta_n\xi_n$, $\xi_n \in Tu_n$, $\forall n \geq 0$ … (10)

Remark 2: T is strongly (pseudo) contraction $\Leftrightarrow (I - T)$ is strongly (accretive) and verse versa.

Theorem 8: Let μ_γ be a uniformly smooth space, $\emptyset \neq A \subset \mu_\gamma$, A be a convex and bounded subset of μ_γ, and $T: A \to 2^A$ be a Lipschitz strongly pseudo contraction mapping with Lipschitz constant $L \geq 1$. Suppose that $F(T) \neq \emptyset$, let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in (0,1) such that :

(i) $\alpha_n + \beta_n = 1$
(ii) $\lim_{n\to\infty}\beta_n = 0$
(iii) $\sum_{n=1}^{\infty}\alpha_n = \infty$

Then for any $u_0 \in A$, the sequence $\{u_n\}$ in (10)

Proof: Let $p \in F(T)$. From the condition (10) and proposition (2) $\Rightarrow$ $u_{n+1} - p = (1-\alpha_n)(u_n - p) + \alpha_n(\mu_n - p)$, where $\mu_n \in Tv_n$

$\gamma^2(u_{n+1} - p) = \gamma^2((1-\alpha_n)(u_n - p) + \alpha_n(\mu_n - p))$
$\leq (1 - 2\alpha_n)\gamma^2(u_n - p) + 2\alpha_n\langle \mu_n - p | h(u_n - p)\rangle + 2\alpha_n\varepsilon$ ... (11)

Using Lemma (2), Remark (2) and Condition (4), to obtain

$\langle \mu_n - p | h(u_n - p)\rangle = \langle \mu_n - \xi_n + \xi_n - u_n + u_n - p | h(u_n - p)\rangle$, where $\xi_n \in Tu_n$

$= \langle \mu_n - \xi_n | h(u_n - p)\rangle + \langle u_n - p | h(u_n - p)\rangle - \langle (u_n - \xi_n) - (p - p) | h(u_n - p)\rangle$
$\leq \gamma(\mu_n - \xi_n)\gamma(u_n - p) + \gamma^2(u_n - p) - k\gamma^2(u_n - p) - k\gamma^2(u_n - p)$

Condition (1)and(8) imply that

$\langle \mu_n - p | h(u_n - p)\rangle \leq H(Tv_n, Tu_n)\gamma(u_n - p) + \gamma^2(u_n - p) - k\gamma^2(u_n - p) + b_n\gamma(u_n - p)$
$\leq L\gamma(v_n - u_n)\gamma(u_n - p) + \gamma^2(u_n - p) - k\gamma^2(u_n - p) + b_n A$ … (12)

where $A = \sup\gamma(u_n - p)$

Using conditions (10), (1) and (8), to obtain

$\gamma(v_n - u_n) = \gamma((1-\beta_n)u_n + \beta_n\xi_n - \beta_n p + \beta_n p - u_n)$
$= \gamma(\beta_n(p - u_n) + \beta_n(\xi_n - p))$
$\leq \beta_n\gamma(u_n - p) + \beta_n\gamma(\xi_n - p)$
$\leq \beta_n\gamma(u_n - p) + \beta_n H(Tu_n, Tp) + \beta_n d_n$

Then,

$L\gamma(v_n - u_n)\gamma(u_n - p) = L(\beta_n\gamma(1+L) + L\beta_n\gamma(u_n - p) + \beta_n d_n)\gamma(u_n - p)$
$= L\beta_n\gamma^2(u_n - p) + L^2\beta_n\gamma^2(u_n - p) + L\beta_n d_n\gamma(u_n - p)$
$\leq L\beta_n(1+L)\gamma^2(u_n - p) + L\beta_n d_n A$, .. (13)

where $A = \sup\gamma(u_n - p)$. Substituting (13) in (12), to obtain

$\langle \mu_n - p | h(u_n - p)\rangle \leq L\beta_n(1+L)\gamma^2(u_n - p) + \gamma^2(u_n - p) - k\gamma^2(u_n - p) + b_n A + L\beta_n d_n A$
$\leq (1 - K + L\beta_n(1+L))\gamma^2(u_n - p) + b_n A + L\beta_n d_n A$

Condition (ii) implies that, $\beta_n L(1+L) \leq k(1-k)$, for large n enough so that

$\langle \mu_n - p | h(u_n - p)\rangle \leq (1 - k + k(1-k))\gamma^2(u_n - p) + b_n A + L\beta_n d_n A$

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$= (1 - k^2)\gamma^2 (u_n - p) + b_n A + L\beta_n d_n A$ ... (14)

Substituting (14) in (11) implies that

$\gamma^2(u_{n+1} - p) \le (1 - 2\alpha_n)\gamma^2 (u_n - p) + 2\alpha_n(1 - k^2)\gamma^2 (u_n - p) + 2\alpha_n\varepsilon + b_n A + L\beta_n d_n A$

$= (1 - 2k^2\alpha_n)\gamma^2 (u_n - p) + 2\alpha_n\varepsilon + 2\alpha_n b_n A + 2\alpha_n L\beta_n d_n A$

Finally, $2\alpha_n b_n A + 2\alpha_n L\beta_n d_n A \to 0$ as $n \to \infty$. Denote $a_n = \gamma(u_n - p)$; $\lambda_n = 2k^2\alpha_n \in (0,1)$. It follows that $0 \le \lim_{n\to\infty}\sup a_n \le \varepsilon$ $\lim_{n\to\infty}\sup a_n = 0$ and so $\lim_{n\to\infty} a_n = 0$. Hence, $\lim_{n\to\infty}\gamma(u_n - p) = 0$, i.e., $\lim_{n\to\infty} u_n = p$.

Theorem 9: Let μ_γ be a uniformly smooth space, $\emptyset \ne A \subset \mu_\gamma$, A be a convex and bounded subset of μ_γ and $T: A \to 2^A$ be a strongly pseudo contraction mapping. Assume that $\{a_n\}$, $\{b_n\}$ be two sequences in (0,1)such that

(i) $\alpha_n + \beta_n = 1,\ \forall n \ge 0$

(ii) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$

(iii) $\sum_{n=1}^{\infty}\alpha_n = \infty$

If $F(T) \ne \emptyset$, then the sequence $\{u_n\}$ defined as in (10) converges to a fixed point of T.

Proof: Let $p \in F(T)$. By Condition (9) and Lemma (3), getting that

$u_{n+1} - p = (1 - \alpha_n)(u_n - p) + \alpha_n(\mu_n - p)$, where $\mu_n \in Tv_n$

$\gamma^2 (u_{n+1} - p) = \gamma^2 ((1 - \alpha_n)(u_n - p) + \alpha_n(\mu_n - p))$

$\le (1 - \alpha_n)^2\gamma^2 (u_n - p) + 2\alpha_n\langle\mu_n - p|h(u_n - p)\rangle$

$= (1 - \alpha_n)^2\gamma^2 (u_n - p) + 2\alpha_n\langle\mu_n - p|h(u_{n+1} - p) - h(v_n - p) + h(v_n - p)\rangle$

$= (1 - \alpha_n)^2\gamma^2 (u_n - p) + 2\alpha_n < \mu_n - p, j(v_n - p) > +2\alpha_n a_n$ ... (15)

where $a_n = \langle\mu_n - p|h((u_{n+1} - p) - h(v_n - p)\rangle$. From (15) and Condition (6) the following holds

$\gamma^2 (u_{n+1} - p) \le (1 - \alpha_n)^2\gamma^2 (u_n - p) + 2\alpha_n(1 - k)\gamma^2(v_n - p) + 2\alpha_n a_n$ ... (16)

The next step to prove that $a_n \to 0$ as $n \to \infty$. Because A is bounded set in μ_γ and u_n, μ_n, ξ_n and $p \in A$, where $\xi_n \in Tu_n, u_{n+1} - p - (v_n - p) = (\beta_n - \alpha_n)u_n + \mu_n\alpha_n - \beta_n\xi_n \to 0$. $J(u_{n+1} - p) - h(v_n - p) \to 0$ and so $a_n \to 0$ as $n \to \infty$

By using Condition (10) and Lemma (3), getting the following

$\gamma^2 (v_n - p) = \gamma^2 ((1 - \beta_n)(u_n - p) + \beta_n(\xi_n - p))$

$\le (1 - \beta_n)^2\gamma^2 (u_n - p) + 2\beta_n\langle\xi_n - p|h(v_n - p)\rangle$

$= (1 - \beta_n)^2\gamma^2 (u_n - p) + 2\beta_n\langle\xi_n - p|h(v_n - p) - h(u_n - p) + h(u_n - p)\rangle$

$= (1 - \beta_n)^2\gamma^2 (u_n - p) + 2\beta_n\langle\xi_n - p|h(u_n - p)\rangle + 2\beta_n b_n$ ... (17)

where $b_n = \langle\xi_n - p|h(v_n - p) - h(u_n - p)\rangle$. It is possible to prove $b_n \to 0$ as $n \to \infty$

$\gamma^2 (v_n - p) \le (1 - \beta_n)^2\gamma^2 (u_n - p) + 2\beta_n(1 - k)\gamma^2 (u_n - p) + 2\beta_n b_n$

$\le \gamma^2 (u_n - p) + 2\beta_n(1 - k)A + 2\beta_n b_n$ ... (18)

where $A = \sup\gamma(u_n - p) < \infty$. Substituting (18) in (16), having the following

$\gamma^2 (x_{n+1} - p) \le (1 - \alpha_n)\gamma^2 (u_n - p) + 2\alpha_n(1 - k)\{\gamma^2 (u_n - p) + 2\beta_n(1 - k)A + 2\beta_n b_n\} + 2\alpha_n a_n$

$= \{(1 - k\alpha_n + \alpha_n(\alpha_n - k)\}\gamma^2 (u_n - p) + \alpha_n c_n$,

where $c_n = 4(1 - k)\{\beta_n(1 - k)A + \beta_n b_n\} + 2a_n$. Because $\alpha_n \to 0$ as $n \to \infty$,

$\gamma^2 (x_{n+1} - p) \le (1 - k\alpha_n)\gamma^2 (u_n - p) + \alpha_n c_n$,

$a_n = \gamma(u_n - p)$, $\lambda_n = k\alpha_n \in (0,1)$ and $\sigma_n = \alpha_n c_n$. Hence, $\lim_{n\to\infty} a_n = 0$. Which implies that $\lim_{n\to\infty}\gamma(u_n - p) = 0$, thus $\lim_{n\to\infty} u_n = p$.

Lemma 4 (15)**:** Let $\{a_n\} \subset [0,\infty)$ such that $a_{n+1} \le (1 - \omega)a_n + \sigma_n\delta$, where $\omega \in (0,1), \delta > 0$ are fixed numbers$\sigma_n \ge 0, \forall n \in N, \lim_{n\to\infty}\sigma_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

Theorem 10: Let μ_γ be a real modular space, A be a non-empty convex and bounded subset of μ_γ and $T: A \to 2^A$ be a multi-valued strongly pseudo contraction mapping. Assume that (α_n), (β_n) sequences in (0,1) and $\alpha_n + \beta_n = 1, \forall n$. If $\lim_{n\to\infty}\gamma(\mu_n - \xi_{n+1}) = 0$, where $\mu_n \in Tv_n$ and $\xi_{n+1} \in T u_{n+1}$, $F(T) \ne \emptyset$ and $k \in \left(0, \frac{1}{2}\right)$, then for any given $u_0 \in A$, $\{u_n\}$ converges to a fixed point of T.

Proof: Let $p \in F(T)$. Using condition (10) and Lemma (3), to get

$u_{n+1} - p = (1 - \alpha_n)(u_n - p) + \alpha_n(\mu_n - p)$, where $\mu_n \in Tv_n$

$\gamma^2 (u_{n+1} - p) = \gamma^2 ((1 - \alpha_n)(u_n - p) + \alpha_n(\mu_n - p))$

$\le (1 - \alpha_n)^2\gamma^2(u_n - p) + 2\alpha_n\langle\mu_n - p|h(u_{n+1} - p)\rangle$

$= (1 - \alpha_n)^2\gamma^2(u_n - p) + 2\alpha_n\langle\mu_n - \xi_{n+1} + \xi_{n+1} - p|h(u_{n+1} - p)\rangle$

$\le (1 - \alpha_n)^2\gamma^2(u_n - p) + 2\alpha_n\langle\mu_n - \xi_{n+1}|h(u_{n+1} - p)\rangle + 2\alpha_n\langle\xi_{n+1} - p|h(u_{n+1} - p)\rangle$

Lemma (2) and Condition (6)

$\gamma^2\,(u_{n+1}-p)\ \le (1-\alpha_n)^2\,\gamma^2(u_n-p)+2\alpha_n\,\gamma\,(\mu_n-\xi_{n+1})\gamma\,(u_{n+1}-p)+2\,\alpha_n k\ \gamma^2(u_{n+1}-p),\qquad \forall h\,(u_{n+1}-p)\in H\,(u_{n+1}-p)$

$(1-2\alpha_n k)\,\gamma^2\,(u_{n+1}-p)\ \le\ (1-\alpha_n)^2\,\gamma^2(u_n-p)+2\alpha_n\gamma\,(\mu_n-\xi_{n+1})\gamma\,(u_{n+1}-p)$

$\gamma^2\,(u_{n+1}-p)\ \le \frac{(1-\alpha_n)^2}{(1-2\alpha_n k)}\gamma^2(u_n-p)+\frac{2\alpha_n}{(1-2\alpha_n k)}\,\gamma\,(\mu_n-\xi_{n+1})\gamma\,(u_{n+1}-p)$

Because of $k\in\left(0,\frac{1}{2}\right)$, $\alpha_n\in(0,1)$, $\frac{2(1-k)-\alpha_n}{1-2\alpha_n k}\ \ge 1 \Rightarrow -\left(\frac{2(1-k)-\alpha_n}{1-2\alpha_n k}\right)\ \le\ -1$, so

$$\frac{(1-\alpha_n)^2}{(1-2\alpha_n k)}=\frac{1-2\alpha_n+\alpha_n{}^2}{(1-2\alpha_n k)}=\frac{((1-2\alpha_n k)+2\alpha_n k-2\alpha_n+\alpha_n{}^2}{(1-2\alpha_n k)}$$
$$=1-\left(\frac{2\,(1-k)-\alpha_n}{(1-2\alpha_n k)}\right)\alpha_n\ \le 1-\alpha_n \quad \ldots \quad (19)$$

Now, denote $d=\sup\{\gamma\,(\xi):\xi\ \in Tu\,,u\ \in A\}+\gamma\,(p)$. Because Rang($T$) is bounded, then $d\ <\ \infty$. Denote $A=d+\ \gamma(\,u_0-p\,)+1$. Observe that

$$\gamma\,(u_1-p)=\ \gamma\,((1-\alpha_0)(u_0-p)+\alpha_0(\mu_0-p))$$
$$\le\ (1-\alpha_0)\gamma(u_0-p)+\alpha_0(\,\gamma\,(\,\mu_0)+\ \gamma\,(p)\,)$$
$$\le\ (1-\alpha_0)\,A+\ \alpha_0\,A=A$$

Supposing $\gamma\,(u_n-p\,)\ \le A$, to prove that $\gamma\,(\,u_{n+1}-p\,)\ \le A$. Indeed,

$$\gamma\,(u_{n+1}-p\,)\le(1-\alpha_n)\gamma(u_n-p)+\alpha_n\,\gamma\,(\,\mu_n-p)$$
$$\le (1-\alpha_n)\,A+\ \alpha_n\,(\,\gamma\,(\mu_n)+\ \gamma\,(p)\,)$$
$$\le(1-\alpha_n)\,A+\ \alpha_n\,A=A$$

Thus, getting $\exists\,A\ >0,\gamma\,(\,u_{n+1}-p\,)\ \le A$, $\forall\,n\ \ge 0$. Condition (19) implies that

$$\gamma^2\,(u_{n+1}-p\,)\ \le\ (1-\alpha_n)\,\gamma^2(u_n-p)+\gamma\,(\mu_n-\ \xi_n\,)\frac{2\alpha_n}{(1-2\alpha_n k)}\ A.$$

But $(1-\alpha_n)\ \le(1-\ \omega)$, *and* $\frac{2\alpha_n}{(1-2\alpha_n k)}\ \le\ \frac{2}{(1-2k)}$. So, $\gamma^2\,(u_{n+1}-p\,)\ \le\ (1-\ \omega)\,\gamma^2(u_n-p)+\ \gamma\,(\,\mu_n\ -\ \xi_n)\frac{2}{(1-2k)}\ A$. Denote $a_n=\gamma^2(u_n-p),\sigma_n=\gamma\,(\,\mu_n\ -\ \xi_n)$ and $\delta_n\ =\frac{2}{(1-2k)}\ A$. So, $\lim_{n\to\infty}a_n=0$, which implies that $\lim_{n\to\infty}\gamma(u_n-p)=0$, thus $\lim_{n\to\infty}u_n=p$.

Later, in this direction, it is possible to study the results in (13-15) in modular spaces and compression all results.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

References:

1. Nakano H. Modular semi-ordered spaces. Tokyo Mathematical Book Series. Tokyo: Maruzen Co. Ltd. 1950; 288p.
2. Musielak J, Orlicz W. On modular spaces. Stud. Math. MR 0101487. 1959; 18:49-65.
3. Khan SH, Abbas M. Approximating fixed points of multivalued ρ-nonexpansive mappings in modular function spaces. Fix. Poi. Theo. and Appl. 2014; 34: 1-12.
4. Khan SH, Abbas M, Ali S. Fixed point approximation of multivalued ρ-quasi-nonexpansive mappings in modular function spaces. J. of Non. Sci. and Appl. 2017; 10(6): 3168–3179.
5. Abbas M, Rhoades BE. Fixed point theorems for two new classes of multivalued mappings. Appl. Math. Let. 2009; 22(9) :1364–1368.
6. Abduljabbar MF, Abed SS. The convergence of iteration scheme to fixed points in modular spaces. Iraq. J. of Sci. 2019; 60(10): 2197-2202. http://scbaghdad.edu.iq/eijs/index.php/eijs/article/view/1037
7. Abed SS. On invariant best approximation in modular spaces. Glob. J. of Pur. and Appl. Math. 2017; 13 (9): 5227-5233. https://www.ripublication.com/gjpam17/gjpamv13n9_102.pdf
8. Abed SS, Abduljabbar MF. Approximating fixed points in modular spaces. KIJMS. 2020; 6 (2): 121-128. https://kijoms.uokerbala.edu.iq/home/vol6/iss2/3/
9. Latif ShL, Abed SS, . Types of fixed points of set-valued contraction mappings for comparable elements. Ira. J.Sci, 2020; Special Issue: 190-195. http://scbaghdad.edu.iq/eijs/index.php/eijs/article/view/2245
10. Agarwal RP, O'Regan D, Sahu DR. Fixed point theory for Lipschitzian-type mapping with applications. New York: Springer;2009. 368p.
11. Takahashi w. Nonlinear functional analysis, fixed point theory and its applications. Japan: Yokohama Publishers Inc; 2000. 220p.
12. Abed SS, Approximating fixed points of the general asymptotic set-valued mappings. JAM.2020; 18:52-59 (2020), DOI: https://doi.org/10.24297/jam.v18i.8549
13. Abed SS, Mohamed Hasan ZM. Common fixed point of a finite-step iteration algorithm under total asymptotically quasi-nonexpansive maps. BSJ. 2019; 16(3): 654-660. http://dx.doi.org/10.21123/bsj.2019.16.3.0654
14. Abed SS, Taresh NS. On stability of iterative sequences with an error. Mat. 2019; 7 (765): 1-12. https://www.mdpi.com/2227-7390/7/8/765
15. Luaibi HH, Abed SS. Fixed point theorems in general metric space with an application. BSJ.2021: 18 (1):812–815. http://dx.doi.org/10.21123/bsj.2021.18.1(Suppl.).0812

بعض النتائج حول التطبيقات الثنائية المنظمة و النقاط الصامدة التقريبية في فضاءات الوحدات المحدبة الحقيقية

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الخلاصة:
لقد تم تقديم التطبيقات الثنائية المنظمة في فضاءات الوحدات المحدبة الحقيقية ثم برهنا بعض خواصها التي تسمح بنتائج ذات صلة بمفهوم التحدب الناعم النتظم لفضاءات الوحدات المحدبة الحقيقية. تم برهنة تقارب متتابعة ذات خطوتين الى نقطة صامدة لتطبيقات متعددة القيم معرفة على هذه الفضاءات.

الكلمات المفتاحية: تحدب منتظم، تطبيقات متعددة القيم، فضاءات الوحدات الحقيقية، نقاط الصامدة.