Open Access
Baghdad Science Journal **Vol.15(4)2018**

DOI: http://dx.doi.org/10.21123/bsj.2018.15.4.0472

On Preradical of Semimodules

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Received 15/5/2018, Accepted 2/10/2018, Published 6/12/2018

Abstract:

In this paper we study the notion of preradical on some subcategories of the category of semimodules and homomorphisms of semimodules.

Since some of the known preradicals on modules fail to satisfy the conditions of preradicals, if the category of modules was extended to semimodules, it is necessary to investigate some subcategories of semimodules, like the category of subtractive semimodules with homomorphisms and the category of subtractive semimodules with ҽk-regular homomorphisms.

Keywords: ҽk-regular homomorphism, preradical, Ş-semimodule, subtractive Ş-subsemimodule.

Introduction:

Throughout this work, Ş stands for a commutative semiring with identity and a semimodule means a unitary left Ş-semimodule. An Ş-subsemimodule L of an Ş-semimodule A is called subtractive if for all $a, a' \in A$, $a + a', a \in L$ implies $a' \in L$ it is clear that 0 and A are subtractive Ş-subsemimodules of A. An Ş-semimodule A is a subtractive Ş-semimodule if it has only subtractive subsemimodules (1). The category of Ş-semimodules will be denoted by $M_Ş$.

In this work we will introduce the concept of ҽk-regular homomorphism (which are relevant to our work). A homomorphism of Ş-semimodules $\varphi: A \longrightarrow B$ is said to be epimorphism-kernel regular (ҽk-regular) if φ is an epimorphism and ҡ-regular.

In module theory, a preradical Ᵽ on R-Mod (the category of unitary left R-modules) is any subfunctor of the identity functor of R-Mod (2). The concept of preradical in semimodules was not found in the literatures.

In this paper we study the notion of preradical on some subcategories of the category of semimodules and homomorphism of semimodules. Since some of the known preradicals on modules fail to satisfy the conditions of preradicals, if the category of modules was extended to semimodules, it is necessary to investigate some subcategories of semimodules, like $M_{ҽk-Ş}$ (the category of subtractive semimodules with ҽk-regular homomorphisms) and the category of subtractive semimodules with homomorphisms $M_{cŞ}$.

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That is only we need to prove some properties of homomorphisms on Ş-semimodule.

In addition to section One, there are five more sections. Section Two consists of the preliminaries that we need in our investigation; some of these were found in the literatures, while the results (Lemma 2.10, which are relevant to our work, and other results in the next sections) are part of our investigation. In section Three, three types of preradicals were investigated, Jacobson radical, socle, and second of semimodule. Singular and torsion of semimodule were studied in Section Four . In Section Five the concepts of reject and inject of semimodule were investigated. Finally, in Section Six, preradicals and their properties and types were introduced applying these concepts to the examples of the previous sections.

Preliminaries

Some definitions and propositions that needed in this work will be introduced.

Definition 2.1(3). Let Ş be a semiring. A left Ş-semimodule $A_Ş$ is a commutative monoid $(A, +, 0)$ for which we have a function $Ş \times A \longrightarrow A$, defined by $(s, a) \mapsto sa$ such that $\forall\, s, t \in Ş$ and $a, a' \in A$,

1- $s(a + a') = sa + sa'$.
2- $(s + t)a = sa + ta$.
3- $(st)a = s(ta)$.
4- $0_Ş\, a = 0_A = s0_A$.

If $1_Ş a = a$ holds then a left Ş-semimodule A is called unitary.

Definition 2.2(3). Let L be a subset of a left Ş-semimodule A then L is called subsemimodule of A if L is closed under addition and scalar multiplication. In this case it is denoted by $L \hookrightarrow A$.

Definition 2.3(3). An Ş-subsemimodule L of an Ş-semimodule A is called subtractive if for all $a, a' \in A$ $a, a + a' \in L$ implies $a' \in L$.

An Ş-semimodule A is a subtractive Ş-semimodule if it has only subtractive subsemimodules.

Definition 2.4(3). Let A and B be Ş-semimodules. A homomorphism from A to B is a map $\varphi: A \longrightarrow B$ such that

1- $\varphi(a + a') = \varphi(a) + \varphi(a')$ and
2- $\varphi(sa) = s\varphi(a)$ $\forall\, a, a' \in A$ and $s \in Ş$.

For a homomorphism of Ş-semimodules $\varphi : A \longrightarrow B$ we define:

1- $\ker(\varphi) = \{a \in A | \varphi(a) = 0\}$.
2- $\varphi(A) = \{\varphi(a) | a \in A)\}$.
3- $Im(\varphi) = \{b \in B \mid b + f(a) = f(a')$ for some a, $a' \in A\}$.

A homomorphism of Ş-semimodules $\varphi : A \longrightarrow B$ is a

1- Monomorphism, if for any Ş-semimodule L and Ş-homomorphisms $\alpha, \beta: L \to A$ with $\varphi \circ \alpha = \varphi \circ \beta$ we have $\alpha = \beta$.
2- Epimorphism, if for any Ş-semimodule K and Ş-homomorphisms $\alpha, \beta: B \to K$ with $\alpha \circ \varphi = \beta \circ \varphi$ we have $\alpha = \beta$.
3- Isomorphism if φ is monomorphism and epimorphism.
4- Image regular (i-regular), if $\varphi(A) = Im(\varphi)$.
5- Kernel regular (ƙ-regular) if $\varphi(a) = \varphi(a') \Longrightarrow a + k = a' + k'$ for some, $k, k' \in \ker(\varphi)$.
6- Regular if φ is i-regular and ƙ-regular.

Definition 2.5. A homomorphism of Ş-semimodules $\varphi: A \longrightarrow B$ is said to be epimorphism-kernel regular (ɛƙ-regular) if φ is an epimorphism and ƙ-regular.

Definition 2.6(4). The (possibly infinite) sequence of Ş-semimodules $\{A_i\}_{i \in I}$ $\ldots A_i \xrightarrow{\varphi_i} A_{i+1} \xrightarrow{\varphi_{i+1}} A_{i+2} \xrightarrow{\varphi_{i+2}} \ldots$ is said to be:

1- Exact, if $Im\varphi_i = ker\varphi_{i+1} \forall\, i \in I$.
2- proper exact, if $\varphi_i(A_i) = ker\varphi_{i+1} \forall\, i \in I$.

Proposition 2.7(3). Let A *and* B be Ş-semimodules, then a homomorphism of Ş-semimodules $\varphi: A \longrightarrow B$ is:

1- Injective if and only if is a monomorphism.
2- Surjective if and only if is an epimorphism and $\varphi(A) \subseteq B$ is subtractive.

Lemma 2.8 (3). Let A, B be Ş-semimodules and $\varphi \in Hom_Ş(A, B)$, then

1- $Im(\varphi)$ is subtractive.
2- $\varphi(A)$ is subtractive if and only if $\varphi(A) = Im(\varphi)$.

Proposition 2.9 (5). For any ƙ-regular homomorphism $\varphi: A \longrightarrow B$ from a subtractive semimodule A to a semimodule B and any subsemimodule L of A, $\varphi^{-1}(\varphi(L)) = L + ker\varphi$.

The following lemma serves to consider the category of subtractive Ş-semimodules and ɛƙ-regular Ş- homomorphisms.

Lemma 2.10. Let A, B *and* C be Ş-semimodules and $\varphi: A \longrightarrow B$, $\alpha: B \longrightarrow C$ be ɛƙ-regular Ş-homomorphisms then $\alpha\varphi: A \longrightarrow C$ is also ɛƙ-regular.

Proof: Let $\alpha(\varphi(a)) = \alpha(\varphi(a'))$, since α is ɛƙ-regular, then we have

$\varphi(a) + k = \varphi(a') + k'$ where $k,\ k' \in \ker\alpha \subseteq B = \varphi(A)$.

Then, $k, k' \in \varphi(A) \Longrightarrow \varphi(a + x) = \varphi(a' + x')$, for some $x, x' \in A$.

So, $a + x + t = a' + x' + t'$ for some $t, t' \in \ker\varphi$.

Now,since $\ker\varphi \subseteq \ker(\alpha\varphi) \Longrightarrow t, t' \in ker(\alpha\varphi)$, also $x, x' \in \ker(\alpha\varphi)$. On the other hand, it is clear that $\alpha\varphi$ is epimorphism; therefore, $\alpha\varphi$ is ɛƙ-regular. □

Note that for any semimodule A, the identity 1_A is an ɛƙ-regular homomorphism.

Jacobson Radical, Socle and Second of Semimodule

In this section we study firstly, the Jacobson radical of semimodule. Secondly, we study the socle radical of semimodule. Also, we proved some properties of it depending on some definitions and results. Finally, the notion of second module and second radical were introduced in (6), and investigating some properties of them. We extend these notions to semimodules. Although, the notion of second semimodule was given in (7), but we gave a different definition analogous to the definition of (6).

Jacobson Radical of Semimodule

Definition 3.1.1 (8). Let A be an Ş-semimodule and L be a non-zero Ş-subsemimodule of A. We say that L is a small (superfluous) Ş-subsemimodule of A if for every Ş-subsemimodule U of A , $L + U = A$ implies $U = A$. Then we shall denote a small Ş-subsemimodule L of Ş-semimodule A by $L \leqslant_s A$.

It is known that the image of a small submodule under a homomorphism of modules is small. This is not true, in general for semimodules and homomorphisms of semimodules (see Example(1)(8)). In (8) this property was proved under certain condition on homomorphism. The following lemma uses a weaker condition to prove the same property.

Lemma 3.1.2 Let A, B be subtractive Ş-semimodules, $L \leqslant_s A$ and let $\varphi \in Hom_Ş(A, B)$, if φ is ɛƙ-regular. Then $\varphi(L) \leqslant_s B$.

Proof: Assume that $\varphi(L) + U = B$, where $U \hookrightarrow B$.

Now, Let $a \in A \Longrightarrow \varphi(a) = \varphi(l) + u$, for some $l \in L \wedge u \in U$.

Since $\varphi(A)$ is subtractive, it follows $u \in \varphi(A)$ and $u = \varphi(x)$ for some $\in A$ $(x \in \varphi^{-1}(U))$.
Then $\varphi(a) = \varphi(l + x)$. Since φ is ɬk-regular, we obtain
$a + k = l + x + k'$, where, $k' \in Ker\varphi \subseteq \varphi^{-1}(U) \Longrightarrow \quad a \in L + \varphi^{-1}(U)$. Thus $A = L + \varphi^{-1}(U)$.
Since $L \leqslant_s A$, it follows $\varphi^{-1}(U) = A$. Thus $U = B$ (φ is an epimorphism). Hence $\varphi(L) \leqslant_s B$. □

Recall that a maximal Ş-subsemimodule of a semimodule A is a proper subsemimodule of A that is not contained properly in any other proper subsemimodule of A.

Definition 3.1.3(8). Let A be an Ş-semimodule, the Jacobson radical of A is denoted by $J(A)$ and defined as

$$J(A) = \bigcap_{K \subseteq A} \{K : K \text{ is a maximal subsemimodule of } A\}.$$

Proposition 3.1.4(8). Let A be an Ş-semimodule, then

$$\sum_{L \leqslant_s A} L = \bigcap_{K \subseteq A} \{K : \text{is a maximal subsemimodule of } A\}.$$

Proposition 3.1.5. Let A be an Ş-semimodule then:
1- $J(A) \hookrightarrow A$.
2- Let A and B be subtractive Ş-semimodules and $\varphi \in Hom_Ş(A, B)$ be ɬk-regular Ş-homomorphism, then $\varphi(J(A)) \subseteq J(B)$.

Proof: (1) it is obvious by Definition(3.1.3).
(2) by Proposition(3.1.4) $J(A) = \sum_{L \leqslant_s A} L$.
Then $\varphi\big(J(A)\big) = \varphi\big(\sum_{L \leqslant_s A} L\big) = \sum_{L \leqslant_s A} \varphi(L)$,
Since φ is ɬk-regular, so by the Lemma(3.1.2).
$\varphi\big(J(A)\big) = \varphi\big(\sum_{L \leqslant_s A} L\big) = \sum_{\varphi(L) \leqslant_s B} \varphi(L) \subseteq \sum_{K \leqslant_s B} K = J(B)$. □

Note that the previous result (2), was proved in (8), with condition on φ, ɬ-quasiregular, but we proved the result with the weaker condition ɬ-regular.

Lemma 3.1.6. Let A be an Ş-semimodule, Ũ is maximal in A/C if and only if $U = \nu^{-1}(Ũ)$ is maximal in A.

Proof: ⇛ Assume that $\nu: A \longrightarrow A/C$ be an Ş-homomorphism and. Ũ is maximal in A/C, then $U = \nu^{-1}(Ũ) \leftrightarrow Ũ = U/C$. To prove U is maximal in A.
Now, let $U \subseteq V \subseteq A \Longrightarrow Ũ \subseteq V/C \subseteq A/C$ for some subsemimodule V of A containing C.
Then $V/C = Ũ$ or $V/C = A/C$ (since Ũ is maximal), That is $V = U$ or $V = A$.
Hence U is a maximal in A.

⇚ Assume that $U = \nu^{-1}(Ũ)$ is maximal in A and let $Ũ \subseteq \tilde{V} \subseteq A/C$, but $\tilde{V} = V/C$ for some subsemimodule V of A containing C.
So, $U/C \subseteq V/C \subseteq A/C$, which implies $U \subseteq V \subseteq A$.
Since U is maximal in A,
Thus $V = U$ or $V = A$, that is $Ũ = \tilde{V}$ or $\tilde{V} = A/C$.
Hence Ũ is a maximal in A/C. □

Proposition 3.1.7. Let A be an Ş-semimodule and J is a Jacobson radical of A then $J(A/J(A)) = 0$.

Proof: Let
$J(A/J(A)) =$
$\bigcap\{Ũ | Ũ \text{ is maximal subsemimodule in } A/J(A)\}$
and $J(A) = C$.
Since Ũ is maximal subsemimodule in $A/J(A) \Longrightarrow Ũ = U/C$, (where U is a maximal subsemimodule in A and containing C by Lemma (3.1.6)).
Then
$J(A/J(A)) =$
$\bigcap\{Ũ | Ũ \text{ is maximal subsemimodule of } A/J(A)\}$

$$= \bigcap \{\tfrac{U}{J(A)} \Big| U \text{ is maximal subsemimodule of } A \text{ and containing } J(A)\}$$
$$= \bigcap U/J(A)$$
$$= J(A)/J(A) = 0. \quad \square$$

Socle Radical of Semimodule

Recall that a non-zero Ş-semimodule A is called simple if it has no proper non-zero Ş-subsemimodule.

Definition 3.2.1(5). Let A be an Ş-semimodule. The Socle radical of A is denoted by $\mathcal{S}oc(A)$ and defined as
$\mathcal{S}oc(A) = \sum\{L : L \text{ is a simple Ş-subsemimodule of } A\}$.

Lemma 3.2.2. Let A and B be subtractive Ş-semimodules and $\varphi \in Hom_Ş(A, B)$ be ɬk-regular Ş-homomorphism, if L is a simple Ş-subsemimodule of A then $\varphi(L)$ is simple Ş-subsemimodule of B.

Proof: Let L be simple Ş-subsemimodule of A .
If $0 \neq U \hookrightarrow \varphi(L) \Longrightarrow$ either $\varphi^{-1}(U) = 0 \Longrightarrow U = 0$ (which is a contradiction).
Or $\varphi^{-1}(U) = L \Longrightarrow U = \varphi(L)$ (since φ is an epimorphism).
Then $0 \neq \varphi^{-1}(U) \hookrightarrow \varphi^{-1}(\varphi(L))$.
Now, since φ is ɬ-regular, so by Proposition (2.9) we have
$\varphi^{-1}(U) \hookrightarrow L + Ker\varphi \Longrightarrow \varphi^{-1}(U) \hookrightarrow L$ (which is a contradiction). □

Proposition 3.2.3. Let A be an Ş-semimodule then:
1- $\mathcal{S}oc(A) \hookrightarrow A$.
2- Let A and B be subtractive Ş-semimodules and $\varphi \in Hom_Ş(A, B)$,if φ is ɬk-regular then $\varphi(\mathcal{S}oc(A)) \subseteq \mathcal{S}oc(B)$.

Proof: (1) it is clear by Definition(3.2.1).

(2) by Definition(3.2.1), $\mathcal{S}oc(A) = \sum\{L : L$ is a simple Ş-subsemimodule of $A\}$.
Then,$\varphi(\mathcal{S}oc(A)) = \varphi(\sum L) = \sum \varphi(L)$
(L is a simple Ş-subsemimodule of A).
Since φ is ƙ-regular, so by Lemma (3.2.2), we have
$\varphi(\mathcal{S}oc(A)) = \sum\{\varphi(L) : L$ is a simple Ş-subsemimodule of $A\}$
$= \sum\{\varphi(L): \varphi(L)$ is a simple Ş-subsemimodule of $B\}$
$\subseteq \sum\{K: K$ is a simple Ş-subsemimodule of $B\} = \mathcal{S}oc(B)$. □

Second Radical of Semimodule:

Recall that a left annihilator of x in Ş is defined by $ann_Ş(x) = \{s \in Ş | sx = 0\}$, it is clear that $ann_Ş(x)$ is a left ideal of Ş. Also if L is a subsemimodule of A, then $ann_Ş(L) = \{s \in Ş | sx = 0 \ \forall \ x \in L\}$.

Definition 3.3.1 An Ş-semimodule A is called a second semimodule where $A \neq 0$ and $ann_Ş(A) = ann_Ş(A/L) \ \forall L \subseteq A$. Any subsemimodule of A is said to be second if it is a second semimodule.

Definition 3.3.2. Let A be an Ş-semimodule. The Second radical of A is denoted by $\mathcal{S}ec(A)$ and defined as
$\mathcal{S}ec(A) = \sum\{L : L$ is a second Ş-subsemimodule of $A\}$.

Lemma 3.3.3. Let A and B be subtractive Ş-semimodules and $\varphi \in Hom_Ş(A, B)$ be ƙ-regular, if L is a second Ş-subsemimodule of A then $\varphi(L)$ is second Ş-subsemimodule of B.

Proof: Let L be second Ş-subsemimodule of $A \Longrightarrow ann_Ş(L) \supseteq ann_Ş(L/K), \quad \forall K \hookrightarrow L \ \wedge \ K \neq L.$
And let $K' \hookrightarrow \varphi(L) \ \wedge \ s\varphi(L) \subseteq K' \quad \forall s \in Ş.$
Now $s\varphi(L) \subseteq K' \Longrightarrow \varphi(sL) \subseteq K' \quad \forall s \in Ş.$
So, $\varphi^{-1}(\varphi(sL)) \subseteq \varphi^{-1}(K') \Longrightarrow sL + ker\varphi \subseteq \varphi^{-1}(K')$ (by Proposition(2.9)).
We have $sL \subseteq K \Longrightarrow sL = 0 \quad \forall s \in Ş.$
Thus $sL \subseteq \varphi^{-1}(K') \Longrightarrow sL = 0 \Longrightarrow s\varphi(L) = 0.$
Hence $\varphi(L)$ is second Ş-subsemimodule of B. □

Proposition 3.3.4. Let A be an Ş-semimodule then:
1- $\mathcal{S}ec(A) \hookrightarrow A$.
2- Let A and B be subtractive Ş-semimodules and $\varphi \in Hom_Ş(A, B)$, if φ is ƙ-regular then $\varphi(\mathcal{S}ec(A)) \subseteq \mathcal{S}ec(B)$.

Proof: (1) it is clear by Definition(3.3.2).
(2) by Definition(3.3.2), $\mathcal{S}ec(A) = \sum\{L : L$ is a second Ş-subsemimodule of $A\}$
Then,$\varphi(\mathcal{S}ec(A)) = \varphi(\sum L) = \sum \varphi(L)$ (is a second Ş-subsemimodule of A)
Since φ is ƙ-regular, so by Lemma (3.3.3), we have
$\varphi(\mathcal{S}ec(A)) = \sum\{\varphi(L): L$ is a second Ş-subsemimodule of $A\}$
$= \sum\{\varphi(L): \varphi(L)$ is a second Ş-subsemimodule of $B\}$
$\subseteq \sum\{K: is$ a second Ş-subsemimodule of $B\} = \mathcal{S}ec(B)$. □

Singular and Torsion of Semimodule

In this section, we study two concepts singular subsemimodule and torsion subsemimodule and some characterization of them.

Singular of Semimodule

Definition 4.1.1(9): An ideal I of a semiring Ş is said to be an essential ideal of Ş if for every ideal K of Ş , $I \cap K = 0$ implies $K = 0$. Then we shall denote an essential ideal I of a semiring Ş-semimodule by $I \trianglerighteq_e Ş$.

Definition 4.1.2(9): Let A be an Ş-semimodule, a singular subsemimodule of A is denoted by $\mathcal{Z}(A)$ and defined as $\mathcal{Z}(A) = \{a \in A | ann_Ş(a) \trianglerighteq_e Ş\}$.We say that A is singular if $\mathcal{Z}(A) = A$ and nonsingular if $\mathcal{Z}(A) = 0$.

Proposition 4.1.3. Let A be an Ş-semimodule and L be Ş-subsemimodule of A then:
1- $\mathcal{Z}(A) \hookrightarrow A$.
2- $\mathcal{Z}(L) = L \cap \mathcal{Z}(A)$.
3- $\mathcal{Z}(\mathcal{Z}(A)) = \mathcal{Z}(A)$.
4- Let A and B be Ş-semimodules, if $\varphi \in Hom_Ş(A, B)$ then $\varphi(\mathcal{Z}(A)) \subseteq \mathcal{Z}(B)$.

Proof: (1),(2) and (3) they are clear by Definition(4.1.2).
(4) let $a \in \mathcal{Z}(A) \Longrightarrow ann_Ş(\varphi(a)) \supseteq ann_Ş(a) \trianglerighteq_e Ş \Longrightarrow ann_Ş(\varphi(a)) \trianglerighteq_e Ş$. Thus $\varphi(a) \in \mathcal{Z}(B)$.
Hence $\varphi(\mathcal{Z}(A)) \subseteq \mathcal{Z}(B)$. □

Torsion of Semimodule

Definition 4.2.1(10): Let A be an Ş-semimodule. An element $a \in A$ is said to be torsion element if there exists $0 \neq s \in Ş$ such that $sa = 0$. A torsion subsemimodule of A is denoted by $\mathcal{T}(A)$ and define as $\mathcal{T}(A) = \left\{a \in A \middle| \begin{matrix} sa = 0 \\ \text{for some } 0 \neq s \in Ş \end{matrix}\right\}$.We say that A is torsion if $\mathcal{T}(A) = A$ and torsion free if $\mathcal{T}(A) = 0$.

In order to have $\mathcal{T}(A)$ a subsemimodule of A. We need to convert Definition (4.2.1) to the following:

Definition 4.2.2. Let A be a semimodule over a commutative semiring Ş. An element $a \in A$ is said to be torsion element if $sa = 0$, for some nonzero divisor $s \in Ş$. then

$\mathcal{T}(A) =$
$\{a \in A | sa = 0$ for some nonzero divisor $s \in Ş\}$.

Proposition 4.2.3. Let A be an Ş-semimodule and L be Ş-subsemimodule of A then:
1- $\mathcal{T}(A) \hookrightarrow A$.
2- $\mathcal{T}(L) = L \cap \mathcal{T}(A)$.

3- $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$.
4- $\mathcal{T}(A/\mathcal{T}(A)) = 0$.
5- Let A and B be Ş-semimodules, if $\varphi \in Hom_{Ş}(A,B)$ then $\varphi(\mathcal{T}(A)) \subseteq \mathcal{T}(B)$.

Proof: (1),(2) and (3) they are clear by Definition(4.2.1).

(4) Let $0 \neq a + \mathcal{T}(A) \in A/\mathcal{T}(A) \Longrightarrow a \notin \mathcal{T}(A)$

Now, if $s(a + \mathcal{T}(A)) = 0$, then $sa + \mathcal{T}(A) = 0$, so $sa \in \mathcal{T}(A)$ for some nonzero divisor $s \in Ş$.

Hence $t(sa) = 0$, for some nonzero divisor $t \in Ş$ implies $(ts)a = 0$, that is, $a \in \mathcal{T}(A)$ a contradiction.

Therefore $a + \mathcal{T}(A)$ is not torsion. Thus $\mathcal{T}(A/\mathcal{T}(A)) = 0$.

(5) By Definition (4.2.1), $\mathcal{T}(A) =$
$\{a \in A | sa = 0$ for some nonzero divisor $s \in Ş\}$.
Now, let $a \in \mathcal{T}(A) \Longrightarrow sa =$
0 for some nonzero divisor $s \in Ş$.
$\Longrightarrow \varphi(sa) = 0$, for some nonzero divisor $s \in Ş$.
$\Longrightarrow s\varphi(a) = 0$ for some nonzero divisor $s \in Ş$.
Thus $\varphi(a) \in \mathcal{T}(B)$.
Hence $\varphi(\mathcal{T}(A)) \subseteq \mathcal{T}(B)$. □

Reject and Inject of Semimodule

In the following we need to define subcategories of the category of Ş-semimodule and Ş-homomorphism.

1- The category of subtractive Ş-semimodules and Ş-homomorphism, which will be denoted by $M_Ş$.
2- The category of subtractive Ş-semimodules and ɐk-regular (see Lemma 2.10), which will be denoted by $M_{ɐk-Ş}$.

Definition 5.1 (1)(p.54). Let Δ be a class of semimodules. The reject of an Ş-semimodule A denoted by $ℜ_\Delta(A)$ is defined by $ℜ_\Delta(A) = \cap \ker f \{f \in Hom_Ş(A,B), B \in \Delta\}$.

Proposition 5.2 (1)(p.58). Let Δ be a class of all Ş-semimodules. If $A, B \in \Delta$ and $f \in Hom_Ş(A,B)$, then $f(ℜ_\Delta(A)) \subseteq ℜ_\Delta(B)$.

The following operator has an analogous in modules with abbreviation "Trace", we will define it for semimodules using another name and symbol convenient to Definition(5.1).

Definition 5.3. Let Δ be a class of semimodules and A an Ş-semimodule. The inject of A denoted by $ɮ^\Delta(A)$ and defined as $ɮ^\Delta(A) = \sum Imf \{f \in Hom_Ş(B,A),\ B \in \Delta\}$.

Proposition 5.4. Let Δ be a class of all subtractive Ş -semimodules. If $A, B \in \Delta$ and $f \in Hom_Ş(A,B)$, then $f(ɮ^\Delta(A)) \subseteq ɮ^\Delta(B)$.

Proof: Suppose $x \in ɮ^\Delta(A)$and $h \in Hom_ş(L,B)$, $L \in \Delta$then $hf \in Hom_ş(L,A)$
$\Longrightarrow f(x) \in Imh$. (Note that in $M_{cŞ}$ Imh= $h(L)$, see 2.7(2))
Hence $f(ɮ^\Delta(A)) \subseteq ɮ^\Delta(B)$. □

Preradical on Semimodules

This section consists of definitions and examples about preradicals on semimodules analogous to the cases in modules (see (2)), also some results, to solve the problems arises through the extension of the category of modules to that of semimodules, are proved in the category of semimodules or some subcategories of it.

Definition 6.1. A functor $Ᵽ: M_Ş \longrightarrow M_Ş$ is called preradical on $M_Ş$ if the following two conditions are satisfied:

1- $Ᵽ(A) \hookrightarrow A,\ \forall A \in M_ş$.
2- $\forall \varphi \in Hom_Ş(A,B)\ \varphi(Ᵽ(A)) \subseteq Ᵽ(B)$.

In this definition (and what follow) the category$M_Ş$, can be replaced by $M_{cŞ}$ (The subcategory of subtractive Ş-semimodules and Ş-homomorphism) or by $M_{ɐk-Ş}$ (the subcategory of subtractive Ş-semimodules and ɐk-regular homomorphisms).

Examples 6.2.

6.2.1 By Propositions (3.1.5), (3. 2.3) and (3.3.4) the functors $J, \mathcal{S}oc$ and $\mathcal{S}ec$ are preradicals on the category $M_{ɐk-Ş}$.

6.2.2 By Propositions (4.1.3(1,4)) and (4.2.3(1,5)) the functors $\mathcal{Z}$ and $\mathcal{T}$ are preradicals on the category $M_Ş$.

6.2.3 By Proposition (5.2) a functor $ℜ_\Delta$ is preradical on the category $M_Ş$ and by Proposition (5.7) the functor $ɮ^\Delta$ is preradical on the category $M_{cŞ}$.

Definitions 6.3. $\forall A \in M_Ş$, a Preradical $Ᵽ$ is called,

1- Radical if $Ᵽ(A/Ᵽ(A)) = 0$.
2- Idempotent if $Ᵽ(Ᵽ(A)) = Ᵽ(A)$.
3- Hereditary if $Ᵽ(L) = L \cap Ᵽ(A)$, with $L \hookrightarrow A$.

Examples 6.4.

6.4.1 The preradicals J and $\mathcal{T}$ are radicals by Proposition ((3.1.7),(4.2.2)(4))) .

6.4.2 The preradical $\mathcal{S}oc$ is hereditary and idempotent (by (5), Corollary 3.11), and also the preradicals $\mathcal{Z}, \mathcal{T}$ are hereditary and idempotent by Propositions (4.1.3(2,3), (4.2.3(2,3))).

Note 6.5. Let $Ᵽ$ be a preradical $\forall A \in M_ş$. If $\{Ᵽ_i : i \in I\}$ is a collection of preradicals then

1- $\bigcap_{i\in I} Ᵽ_i$, is defined as $(\bigcap_{i\in I} Ᵽ_i)(A) = \bigcap_{i\in I} Ᵽ_i(A)$, is preradical.
2- $\sum_{i\in I} Ᵽ_i$, is defined as $(\sum_{i\in I} Ᵽ_i)(A) = \sum_{i\in I} Ᵽ_i(A)$, is preradical.
3- Let $Ᵽ$, $Ł$ be two preradicals, then $Ᵽ \leq Ł$, if $Ᵽ(A)) \subseteq Ł(A)$, $\forall A \in M_ş$.

Definition 6.6. Let $\mathcal{M}$ be a class of semimodules, $\mathcal{M}$ is said to be,

1- Hereditary if $\mathcal{M}$ is closed under isomorphisms and subsemimodules.
2- Cohereditary if $\mathcal{M}$ is closed under epimorphic images.

Definition 6.7. Let $\mathbb{P}$ be a preradical, an $\mathbb{S}$-semimodule $A \in M_{\mathbb{S}}$ is said to be ,

1- $\mathbb{P}$-torsion if $\mathbb{P}(A) = A$. A class of all $\mathbb{P}$-torsion semimodules is denoted by $\mathbb{T}_{\mathbb{P}}$.
2- $\mathbb{P}$-torsionfree if $\mathbb{P}(A) = 0$. A class of all $\mathbb{P}$-torsionfree semimodules is denoted by $\mathbb{F}_{\mathbb{P}}$.

Proposition 6.8. Let $A \in M_{\mathbb{S}}$ and $\mathbb{P}$ be a preradical , $\forall\, K \hookrightarrow A$,then:

1- $\mathbb{P}(K) \subseteq K \cap \mathbb{P}(A)$.
2- $(\mathbb{P}(A) + K)/K \subseteq \mathbb{P}(A/K)$.
3- If $\mathbb{P}(A/K) = 0\ then\ \mathbb{P}(A) \subseteq K$.
4- If $\mathbb{P}(K) = K\ then\ K \subseteq \mathbb{P}(A)$.

Proof: (1) and (2) by definition of preradical. (3) by (2) and (4) by (1). □

Proposition 6.9. Let $\mathbb{P}$ be a preradical. If $\{ A_i : i \in I\}$ is a collection of $\mathbb{S}$-$semimodules$ then:

1- $\mathbb{P}(\bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} \mathbb{P}(A_i)$.
2- $\mathbb{P}(\prod_{i\in I} A_i) \subseteq \prod_{i\in I} \mathbb{P}(A_i)$.

Proof: Since $\mathbb{P}$ is a subfunctor of the identity functor. □

Proposition 6.10. Let $\mathbb{P}$ be a preradical, then:

1- $\mathbb{T}_{\mathbb{P}}$ is a cohereditary class closed under arbitrary direct sums.
2- $\mathbb{F}_{\mathbb{P}}$ is a hereditary class closed under arbitrary direct products.

Proof: (1). Let $A \in \mathbb{T}_{\mathbb{P}} \Longrightarrow \mathbb{P}(A) = A, \varphi: A \longrightarrow B$ be an epimorphism.

Then $\varphi(\mathbb{P}(A)) \subseteq \mathbb{P}(B) \Longrightarrow \varphi(A) \subseteq \mathbb{P}(B)$ (since $\mathbb{P}(A) = A$)

$\Longrightarrow B \subseteq \mathbb{P}(B)$ (since $\varphi$is onto) ….(i).

But $\mathbb{P}(B) \subseteq B$ (by definition) ….(ii).

So, from (i) and (ii) we have $\mathbb{P}(B) = B \Longrightarrow B \in \mathbb{T}_{\mathbb{P}}$.

Now, let $\{A_i : i \in I\} \subseteq \mathbb{T}_{\mathbb{P}} \Longrightarrow \mathbb{P}(A_i) = A_i\ \forall\ i \in I$.

Then $\mathbb{P}(\bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} \mathbb{P}(A_i) = \bigoplus_{i\in I} A_i \Longrightarrow \bigoplus_{i\in I} A_i \in \mathbb{T}_{\mathbb{P}}$.

Hence $\mathbb{T}_{\mathbb{P}}$ is a cohereditary class closed under arbitrary direct sums.

(2). Let $A \in \mathbb{F}_{\mathbb{P}} \Longrightarrow \mathbb{P}(A) = 0$.

So, if $K \subseteq A \Longrightarrow \mathbb{P}(K) \subseteq K \cap \mathbb{P}(A) \Longrightarrow \mathbb{P}(K) \subseteq K \cap 0 = 0 \Longrightarrow \mathbb{P}(K) = 0$.

Thus $K \in \mathbb{F}_{\mathbb{P}}$.

Now, let $\varphi: A \longrightarrow K$ be an isomorphism and $A \in \mathbb{F}_{\mathbb{P}}$.

Then $\varphi^{-1}\big(\mathbb{P}(K)\big) \subseteq \mathbb{P}(A) = 0 \Longrightarrow \mathbb{P}(K) \subseteq ker\varphi^{-1} = 0$, So $K \in \mathbb{F}_{\mathbb{P}}$.

Now, let $\{ A_i : i \in I\} \subseteq \mathbb{F}_{\mathbb{P}} \Longrightarrow \mathbb{P}(A_i) = 0\ \forall\ i \in I$.

Since $\mathbb{P}(\prod_{i\in I} A_i) \subseteq \prod_{i\in I} \mathbb{P}(A_i) = 0 \Longrightarrow \prod_{i\in I} A_i \in \mathbb{F}_{\mathbb{P}}$

Hence $\mathbb{F}_{\mathbb{P}}$ is a hereditary class closed under arbitrary direct products. □

Proposition 6.11. Where $\mathbb{P}$ is a preradical, then

1- $\mathrm{Hom}_{\mathbb{S}}(T, F) = 0$, $\forall\, T \in \mathbb{T}_{\mathbb{P}}$ and $F \in \mathbb{F}_{\mathbb{P}}$.
2- $\mathbb{T}_{\mathbb{P}} \cap \mathbb{F}_{\mathbb{P}} = 0$.

Proof: (1). Let $T \in \mathbb{T}_{\mathbb{P}}$, $F \in \mathbb{F}_{\mathbb{P}}$ and $\varphi: T \longrightarrow F$, then $\varphi(T) = \varphi(\mathbb{P}(T)) \subseteq \mathbb{P}(F) = 0$. So $\varphi = 0$.

(2). Let $N \in \mathbb{T}_{\mathbb{P}} \cap \mathbb{F}_{\mathbb{P}}$, then $N = \mathbb{P}(N) = 0$. □

Proposition 6.12. Let $\mathbb{P}$ be a radical, then:

1- $T \in \mathbb{T}_{\mathbb{P}}$ if and only if $Hom_{\mathbb{S}}(T, F) = 0$, $\forall\, F \in \mathbb{F}_{\mathbb{P}}$.
2- $\mathbb{T}_{\mathbb{P}}$ is closed under extensions.

Proof: (1). By Proposition (6.11), it is enough to prove the sufficiency.

Let $Hom_{\mathbb{S}}(T, F) = 0$, $\forall\, F \in \mathbb{F}_{\mathbb{P}}$.

Since $T/\mathbb{P}(T) \in \mathbb{F}_{\mathbb{P}}$ ($\mathbb{P}$ is a radical), then the natural map $T \longrightarrow T/\mathbb{P}(T)$ is zero, hence $\mathbb{P}(T) = T \in \mathbb{T}_{\mathbb{P}}$.

2. Follows from 1. □

Proposition 6.13. Let $\mathbb{P}$ be an idempotent Preradical, then:

1- $F \in \mathbb{F}_{\mathbb{P}}$ if and only if $Hom_{\mathbb{S}}(T, F) = 0$, $\forall\, T \in \mathbb{T}_{\mathbb{P}}$.
2- $\mathbb{F}_{\mathbb{P}}$ is closed under extensions.

Proof: (1). By Proposition (6.11), it is enough to prove the sufficiency.

Now, let $Hom_{\mathbb{S}}(T, F) = 0$, $\forall\, T \in \mathbb{T}_{\mathbb{P}}$.

Since $\mathbb{P}(F) \in \mathbb{T}_{\mathbb{P}}(\mathbb{P}$ is an idempotent) , then the inclusion map $\mathbb{P}(F) \longrightarrow F$ is a zero map, hence $\mathbb{P}(F) = 0$.

That is $F \in \mathbb{F}_{\mathbb{P}}$.

(2). Follows from 1. □

Proposition 6.14. Let $\{ A_i : i \in I\}$ be a collection of $\mathbb{S}$-semimodules then:

1- If $A_i \in \mathbb{T}_{\mathbb{P}}\ \forall\ i \in I$, then $\sum_{i\in I} A_i \in \mathbb{T}_{\mathbb{P}}$.
2- If $A/A_i \in \mathbb{F}_{\mathbb{P}}\ \forall\ i \in I$, then $A/\bigcap_{i\in I} A_i \in \mathbb{F}_{\mathbb{P}}$.

Proof: (1) since $A_i \in \mathbb{T}_{\mathbb{P}} \Longrightarrow \mathbb{P}(A_i) = A_i$

Then $\mathbb{P}(\sum_{i\in I} A_i) = \sum_{i\in I} \mathbb{P}(A_i) = \sum_{i\in I} A_i$.

Hence $\sum_{i\in I} A_i \in \mathbb{T}_{\mathbb{P}}$.

(2) since $A/A_i \in \mathbb{F}_{\mathbb{P}} \Longrightarrow \mathbb{P}(A/A_i) = 0$.

For each $j \in I$, let $\varphi_j: A/\cap A_i \longrightarrow A/A_j$ be a natural epimorphism ($a + \cap A_i \mapsto a + A_j$), then

$\varphi_j(\mathbb{P}(A/\cap A_i)) \subseteq \mathbb{P}(A/A_j) = 0$, so $\mathbb{P}(A/\cap A_i) \subseteq \ker \varphi_j = A_j/\cap A_i$ for each j, hence

$\mathbb{P}(A/\cap A_i) \subseteq \cap \ker f_j = \cap A_j/\cap A_i = 0$. Hence $A/\cap A_i \in \mathbb{F}_{\mathbb{P}}$. □

Recall that a functor $\mathcal{F}$ is said to be left exact, if from a given exact sequence $0 \longrightarrow A \xrightarrow{\varphi} M \xrightarrow{\alpha} \mathrm{W} \longrightarrow 0$ we get a left exact sequence $0 \longrightarrow \mathcal{F}(A) \xrightarrow{\mathcal{F}(\varphi)} \mathcal{F}(M) \xrightarrow{\mathcal{F}(\alpha)} \mathcal{F}(\mathrm{W})$.

Proposition 6.15. Let $\mathbb{P}$ be a preradical on $M_{c\mathbb{S}}$, then the following are equivalent:

1- $\mathbb{P}$ is left exact as a functor
2- $\mathbb{P}$ is hereditary
3- $\mathbb{P}$ is idempotent and $\mathbb{T}_{\mathbb{P}}$ is hereditary.

Proof: (1)$\Longrightarrow$(2) let N be a subsemimodule of M then by (1), the sequence

$0 \longrightarrow \mathbb{P}(N) \longrightarrow \mathbb{P}(M) \longrightarrow \mathbb{P}(M/N)$ is exact, so $\mathbb{P}(N) = N \cap \mathbb{P}(M)$.

(2)⟹(3) let P be hereditary, then we have $\mathrm{P}(\mathrm{P}(M)) = \mathrm{P}(M) \cap \mathrm{P}(M) = \mathrm{P}(M)$. So P is an idempotent.

Now, if $K \subseteq M$ and $M \in \mathrm{T}_{\mathrm{P}} \Longrightarrow \mathrm{P}(K) = K \cap \mathrm{P}(M) = K \cap M = K$.

Hence $K \in \mathrm{T}_{\mathrm{P}}$ and T_{P} is hereditary.

(3)⟹(1) let $0 \longrightarrow N \xrightarrow{\varphi} M \xrightarrow{\alpha} \mathrm{W} \longrightarrow 0$ be an exact sequence.

Then the sequence $0 \longrightarrow \mathrm{P}(N) \xrightarrow{\mathrm{P}(\varphi)} \mathrm{P}(M) \xrightarrow{\mathrm{P}(\alpha)} \mathrm{P}(\mathrm{W})$, where $\mathrm{P}(\varphi)$ and $\mathrm{P}(\alpha)$ are restrictions of φ and α. The exactness in $\mathrm{P}(N)$ is clear also the inclusion $Im\mathrm{P}(\varphi) \subseteq ker\mathrm{P}(\alpha)$.

Since $\mathrm{P}(Im\varphi \cap \mathrm{P}(M)) = (Im\varphi \cap \mathrm{P}(M)) \cap \mathrm{P}(\mathrm{P}(M)) = Im\varphi \cap \mathrm{P}(M)$ …(i).

And $\mathrm{P}(Im\varphi \cap \mathrm{P}(M)) \subseteq \mathrm{P}(Im\varphi) = Im\mathrm{P}(\varphi)$ …(ii).

From (i) and (ii) $\Longrightarrow Im\varphi \cap \mathrm{P}(M) \subseteq Im\mathrm{P}(\varphi)$ …(iii).

Now let $a \in ker\mathrm{P}(\alpha)$ be an arbitrary element $\Longrightarrow a \in \mathrm{P}(M) \wedge \alpha(a) = 0 \Longrightarrow a \in \mathrm{P}(M) \wedge a \in Im\varphi$.

Thus $a \in Im\varphi \cap \mathrm{P}(M)$.

So by (i) $a \in Im\mathrm{P}(\varphi) \Longrightarrow ker\mathrm{P}(\alpha) \subseteq ker\mathrm{P}(\varphi)$ as required. □

Theorem 6.16. If P is a Preradical on $M_{c\S}$, then the following are equivalent

1- P is an idempotent radical.
2- $\forall A \in M_{\S}(M_{c\S})$ there exists a uniquely determined (up to isomorphism) proper exact sequence(exact sequence) $0 \to T \to A \to F \to 0$, $F \in \mathrm{F}_{\mathrm{P}}, T \in \mathrm{T}_{\mathrm{P}}$.
3- $\forall A \in M_{c\S}$ there exists an exact sequence(proper exact sequence)$0 \to T \to A \to F \to 0$, $F \in \mathrm{F}_{\mathrm{P}}$, $T \in \mathrm{T}_{\mathrm{P}}$.
4- P is an idempotent radical and T_{P} , F_{P} are closed under extensions.

Proof:(1)⟹(2). Let $0 \longrightarrow \mathrm{P}(A) \longrightarrow (A) \longrightarrow A/\mathrm{P}(A) \longrightarrow 0$ be an exact sequence.

Since P be an idempotent radical $\Longrightarrow \mathrm{P}(A) \in \mathrm{T}_{\mathrm{P}}$.

(2)⟹(3). It is obvious.

(3)⟹(4). Since there is an exact sequence $0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$,

Then $\mathrm{P}(\mathrm{P}(A)) = \mathrm{P}(T) = T = \mathrm{P}(A) \Longrightarrow \mathrm{P}$ is an idempotent preradical with $T \in \mathrm{T}_{\mathrm{P}}$.

Also $F = A/\mathrm{P}(A) \in \mathrm{F}_{\mathrm{P}} \Longrightarrow \mathrm{P}(F) = 0 \Longrightarrow \mathrm{P}(A/\mathrm{P}(A)) = 0 \Longrightarrow \mathrm{P}$ is a radical.

(4)⟹(1). It is obvious. □

Conflicts of Interest: None.

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حول الجذر الابتدائي لشبه المقاسات

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الخلاصة:

في هذا البحث درسنا مفهوم الجذر الابتدائي على بعض الأصناف الجزئية من صنف شبه المقاسات مع تشاكلات شبه المقاسات.

بما ان بعض الجذور الابتدائية المعرفة على صنف المقاسات تفشل في تحقيق شروط الجذر الابتدائي -عند توسيع صنف المقاسات الى صنف شبه المقاسات- فقد كان من الضروري أن نبحث في أصناف جزئية تتحقق فيها شروط الجذر الابتدائي من قبيل صنف شبه المقاسات المطروحة مع تشاكلات شبه المقاسات او صنف شبه المقاسات المطروحة مع تشاكلات شبه المقاسات من النوع ek المنتظمة.

الكلمات المفتاحية: الجذر الابتدائي، شبه المقاس، شبه مقاس -Ş مطروح، تشاكل منتظم-ek.