Existence and Controllability Results for Fractional Control Systems in Reflexive Banach Spaces Using Fixed Point Theorem

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Abstract:
In this paper, a fixed point theorem of nonexpansive mapping is established to study the existence and sufficient conditions for the controllability of nonlinear fractional control systems in reflexive Banach spaces. The result so obtained have been modified and developed in arbitrary space having Opial's condition by using fixed point theorem deals with nonexpansive mapping defined on a set has normal structure. An application is provided to show the effectiveness of the obtained result.

Keywords: Controllability, Fixed point, Fractional control system, Normal structure, Opial's condition.

Introduction:

Many systems in physics, chemistry, biology, stochastic, and control theory are represented by fractional control systems (FCS). For more details on (FCS) in the control theory one can see, Balachandran and Kokila (1), AL-Jawari and Shaker (2), Li Ding and Nieto (3), Lizzy and Balachandran (4).

One of the important topics in the study (FCS) in the control theory is controllability and it means that it is possible to transfer a (FCS) from an arbitrary initial state to an arbitrary final state by using the admissible controls. Thus controllability plays an important role in the analysis and design of these systems, see references (1- 6).

To study the result of the controllability of (FCS), some techniques of nonlinear functional analysis are used such as, fixed point theorems. Lizzy and Balachandran in (4) studied the controllability of stochastic fractional system in Hilbert spaces (HS) by using the Banach contraction mapping theory. Li Ding and Nieto in (3) discussed the controllability of (FCS) using Schauder's fixed point theory.

Since every (HS) are reflexive Banach space (RBS) and the contraction mapping is nonexpansive mapping, but the converse in general is not true (7) (also see section 2 of this paper), thus the purpose of this paper is to study the controllability of (FCS) in arbitrary (RBS) by using fixed point theorem that deals with nonexpansive mapping. The rest of this article is organized as follows. In section 2, preliminaries are given to study the solutions of (FCS) and then to prove the main result (theorem 4) in section 3. In section 4, an application is presented to illustrate the value of the obtained results.

Preliminaries and (FCS):

In this section, the solution of linear and nonlinear (FCS) is explored and present some definitions with theorems that will be used in the prove the main result of controllability (theorem 4) in section 3.

Definition 1 (8): Let Y be a self mapping on a normed space X, such that:
$\|Y(x_1) - Y(x_2)\| \leq \lambda \|x_1 - x_2\|$ for all $x_1, x_2 \in X$. Then Y is contraction if $\lambda < 1$, and Y is nonexpansive if $\lambda \leq 1$.

It can be shown that, a contraction mapping is nonexpansive and isometry mapping is nonexpansive but not contraction, see (8).

Definition 2 (9): Let X be a Banach space such that, if $\forall x \in X$ and $\forall$ sequence $\{x_n\}$ converges weakly to x, then
$\lim_{n\to\infty} inf\|x_n - z\| > \lim_{n\to\infty} inf\|x_n - x\|$, holds $\forall z \neq x$. Thus the space X satisfies Opial's condition.
Every finite dimensional Banach space, (Hilbert space) L_p for $p = 2$ and l_p spaces for $1 < p < \infty$ are satisfies Opial's condition, see (9).

Definition 3 (7): A subset W of a normed space X is called weakly compact if every sequence $\{x_n\}$ in W contains a subsequence which converges weakly in W.

For example, every nonempty, closed, convex and bounded subset of (RBS) is weakly compact see (8-9).

Definition 4 (8): Let W be nonempty, closed, convex and bounded subset of the Banach space X. Let a point $x \in W$, such that $sup\{\|x - w\| : w \in W\} = diam\, W$, then x is called a diametral. Also, W has normal structure, if for each nonempty, convex $K \subseteq W$ with diam $K > 0$, there exist a point $x \in K$ which is not diametral.

Example 1 (8): Every compact convex set in a Banach space has normal structure. And every nonempty, closed, convex and bounded subset of uniformly convex Banach space has normal structure.

Also Opial's condition implies normal structure, see (10).

Theorem 1 (7): Every contraction mapping of a Banach space into itself has a unique fixed point.

Theorem 2 (8, 11): Let Y be nonexpansive from W into W, where W is a nonempty weakly compact convex subset having a normal structure in a Banach space X, then Y has fixed point in W.

Remark 1: The convexity assumption in theorem 2, is very important, because the nonexpansive map on a non-convex set in Banach space may be has no fixed point. For example:

Take $W = [-2, -1] \cup [1, 2] \subset R$ and $Y: W \to W$ by $Y(x) = -x$, $x \in W$, then Y is nonexpansive, but Y has no fixed points in W, see (11).

Here, the solutions of linear and nonlinear (FCS) are discussed. Suppose that, $\eta, \alpha > 0$, with $n - 1 < \eta < n$, $n - 1 < \alpha < n$ and $n \in N$, $[0, h] \subset R$. Let $F = R^n$ and $H = R^m$ be the n and m-dimensional Euclidean spaces.

Throughout this paper, the fractional derivative is taken in the Caputo sense and for brevity let us denote the Caputo fractional derivative by D^η, for more details of properties to D^η, see (12).

Now, consider the linear control system represented by a (FCS) of the form

$$D^\eta z(t) = Oz(t) + Bu(t), t \in I = [0, h],$$
$$z(0) = z_0 \tag{1}$$

where $0 < \eta < 1$, the state vector $z(t) \in F$, the control vector $u(t) \in H$ and O with B are matrices of dimensions $n \times n$, $n \times m$ respectively. The solution of the system Eq.1 can be obtained by using the method of successive approximation, see (12) and given by the following formula

$$z(t) = E_\eta(Ot^\eta)z_0 + \int_0^t (t - s)^{\eta-1} E_{\eta,\eta}(O(t - s)^\eta)Bu(s)ds, \tag{2}$$

where $E_{\eta,\eta}(Ot^\eta) = \sum_{i=0}^\infty \frac{O^i t^{\eta i}}{\Gamma(\eta i + \eta)}$ is the Mittag-Leffler function for a square matrix $O \in R^{n\times n}$, with
$$E_{\eta,1}(Ot^\eta) = E_\eta(Ot^\eta).$$
The function $E_{\eta,\eta}(Ot^\eta)$ is continuous and it satisfies $\|E_{\eta,\eta}(Ot^\eta)\| \leq m$ for all $t \in [0, h]$.

Definition 5 (2) : If $\forall z_0, z_1 \in F, \exists$ a control $u \in L^2([0, h], H)$ such that the solution of the Eq.1 with $z(0) = z_0$ also satisfies $z(h) = z_1$, then saying that the control system Eq.1 is controllable over $[0, h]$.

The control $u(t)$ is said to be admissible control, *i.e.*, $u(t)$ is transfer the trajectory from z_0 to the final state z_1. Thus the controllability of the control system Eq.1 is equivalent to finding $u(t)$ such that

$$z_1 = z(h) = E_\eta(Oh^\eta)z_0 + \int_0^h (h - s)^{\eta-1} E_{\eta,\eta}(O(h - s)^\eta)Bu(s)ds.$$

Equivalently, the system Eq.1 is controllable $\iff \exists$ a control u such that

$$z_1 - E_\eta(Oh^\eta)z_0 = \int_0^h (h - s)^{\eta-1} E_{\eta,\eta}(O(h - s)^\eta)Bu(s)ds \tag{3}$$

The above explanation leads to the following theorem.

Theorem 3 (2): (Controllability Condition) The control system Eq.1 is controllable on $[0, h] \iff$ the controllability Grammian $w(0, h) = \int_0^h (h - s)^{\eta-1} E_{\eta,\eta}(O(h - s)^\eta)BB^* E_{\eta,\eta}(A^*(h - s)^\eta)ds$ is a non-singular, where * denotes the matrix transpose.

Now, consider the nonlinear control system represented by a (FCS) of the form

$$D^\eta z(t) = Oz(t) + Bu(t) + L(t, P(t, z(t))),$$
$$t \in I = [0, h], z(0) = z_0 \tag{4}$$

where η, $z(t)$, $u(t)$, O and B be as defined in the control system Eq.1. The nonlinear operators P and L are continuous from $I \times F$ into $F = R^n$ and satisfy Lipschitz condition on the second argument. For $1 < n \in N$, suppose that Q_n a (RBS) of continuous functions defined from I into F, with norm $\|.\|$. Thus, as above for the control system Eq.1, the solution to control system Eq.4 is given by the following form

$$z(t) = E_\eta(Ot^\eta)z_0 + \int_0^t (t - s)^{\eta-1} E_{\eta,\eta}(O(t - s)^\eta)$$
$$\times Bu(s)ds + \int_0^t (t - s)^{\eta-1} E_{\eta,\eta}(O(t - s)^\eta)$$
$$\times L(s, P(s, z(s)))\, ds \tag{5}$$

Controllability of Nonlinear (FCS):

In this section, the controllability of solution to the nonlinear (FCS) Eq.4 in (RBS) by using fixed point theorem are discussed.

Definition 6: The control system Eq.4 is called controllable on I=[0,h] if $\forall z_0, z_1 \in Q_n$, $\exists$ a control $u \in L^2([0,h], H)$ such that the solution $z(t)$ in Eq.5 satisfies $z(0) = z_0$ and $z(h) = z_1$.

Here, suppose that $w(r) = \{z : z \in Q_n, z(0) = z_0, \|z(t)\| \le r, \forall t \in I\}$ where r is a positive constant. Then w is a closed, convex and bounded subset of Q_n.
Since $w(r)$ is closed subset of a (RBS), then $w(r)$ is a (RBS), see (7).

In order that control system Eq.4 makes sense throughout this paper, let
$$m_1 = sup\|E_\eta(Ot^\eta)z_0\|,$$
$$m_2 = sup\|E_{\eta,\eta}(O(h-t)^\eta)\|$$
and assume the following condition:
$[a_1]$ The nonlinear P and L in Eq.4 are continuous and there exist positive constants m_3, m_4 and m_5 such that for all z_1, $z_2 \in w(r)$ the following inequality hold
$$\|L(t, P(t, z_1(t))) - L(t, P(t, z_2(t)))\| \le m_3\|P(t, z_1(t)) - P(t, z_2(t))\| \le m_3 m_4 \|z_1(t) - z_2(t)\|,$$
also, let $m_5 = \max_{t \in I}\|L(t, P(t, 0))\|$ and
$$m_6 = \max_{t \in I}\|z(t)\|.$$

Theorem 4 : Let Q_n be a (RBS) which satisfying Opial's condition and suppose that the condition $[a_1]$ is satisfied for the nonlinear control system Eq.4, with the control system Eq.1 is controllable on I. Further assume that
$[a_2]$ Let $r = (m_1 + m_2\|B\|k + m_2[m_3 m_4 m_6 + m_5])(\frac{h^\eta}{\eta})$,
where, $k = \|B^*\|m_2\|w^{-1}\|\{\|z_1\|m_1 + m_2[m_3 m_4 m_6 + m_5](\frac{h^\eta}{\eta})\}$
$[a_3]$ Let $\lambda = m_2(\frac{h^\eta}{\eta})m_3 m_4$, such that $0 \le \lambda \le 1$.
Then the nonlinear control system Eq.4 is controllable on the interval $I = [0, h]$.
Proof: Define the operator $Y: Q_n \to Q_n$ as follows
$$(Yz)(t) = E_\eta(Ot^\eta)z_0 + \int_0^t (t-s)^{\eta-1} E_{\eta,\eta}(O(t-s)^\eta)Bu(s)ds + \int_0^t (t-s)^{\eta-1} E_{\eta,\eta}(O(ts)^\eta) \times L(s, P(s, z(s))) \, ds. \quad (6)$$

where the control function $u(t)$ is defined by
$$u(t) = B^* E_{\eta,\eta}(O^*(h-t)^\eta)w^{-1}[z_1 - E_\eta(Oh^\eta)z_0 - \int_0^h (h-s)^{\eta-1} E_{\eta,\eta}(O(h-s)^\eta) \times L(s, P(s, z(s))) \, ds]. \quad (7)$$
The idea of the proof, first to prove that the nonlinear operator Y in Eq.6 is continuous and maps $w(r)$ into itself. Second, to prove Y is nonexpansive mapping from $w(r)$ into $w(r)$ and then by using

theorem 2 the fixed point of the operator Y could obtained.
Thus, by taking the norm of Eq.7 is obtained that
$$\|u(t)\| \le \|B^*\|\|E_{\eta,\eta}(O^*(h-t)^\eta)\|\|w^{-1}\|\{\|z_1\| + \|E_\eta(Oh^\eta)z_0\| + \int_0^h (h-s)^{\eta-1}\|E_{\eta,\eta}(O(h-s)^\eta)\| \times [\|L(s, P(s, z(s))) - L(s, P(s, 0))\| + \|L(s, P(s, 0))\|]ds\}.$$
Then by using condition $[a_1]$ and $m_6 = \max_{t \in I}\|z(t)\|$
is gotten that
$$\|u(t)\| \le \|B^*\|m_2\|w^{-1}\|\{\|z_1\| + m_1 + \int_0^h (h-s)^{\eta-1} m_2[m_3 m_4\|z(t)\| + m_5]ds\}$$
and
$$\|u(t)\| \le \|B^*\|m_2\|w^{-1}\|\{\|z_1\| + m_1 + m_2[m_3 m_4 m_6 + m_5](\frac{h^\eta}{\eta})\} = k$$
Now, define the operator $Y: Q_n \to w(r)$ as in Eq.6 and taking the norm
$$\|(Yz)(t)\| \le \|E_\eta(Ot^\eta)z_0\| + \int_0^t (t-s)^{\eta-1} \times \|E_{\eta,\eta}(O(t-s)^\eta)\|\|B\|\|u(s)\|ds + \int_0^t (t-s)^{\eta-1}\|E_{\eta,\eta}(O(t-s)^\eta)\| \times [\|L(s, P(s, z(s))) - L(s, P(s, 0))\| + \|L(s, P(s, 0))\|] \, ds$$
and then
$$\|(Yz)(t)\| \le m_1 + m_2\|B\|k(\frac{h^\eta}{\eta}) + m_2[m_3 m_4 m_6 + m_5](\frac{h^\eta}{\eta}) = r$$

Since P and L are continuous and $\|(Yz)(t)\| \le r$, it follows that the operator Y is also continuous and maps $w(r)$ into itself.

Second, to show that Y is nonexpansive mapping from $w(r)$ into $w(r)$. Thus for $z_1(t)$, $z_2(t) \in w(r)$ and from the definition $of (Yz)(t)$ in Eq.6 is gotten that

$$\|(Yz_1)(t) - (Yz_2)(t)\| = \|E_\eta(Ot^\eta)z_0 + \int_0^t (t-s)^{\eta-1} E_{\eta,\eta}(O(t-s)^\eta)Bu(s)ds + \int_0^t (t-s)^{\eta-1} E_{\eta,\eta}(O(t-s)^\eta) \times L(s, P(s, z_1(s)))ds - E_\eta(Ot^\eta)z_0 - \int_0^t (t-s)^{\eta-1} E_{\eta,\eta}(O(t-s)^\eta)Bu(s)ds - \int_0^t (t-s)^{\eta-1} E_{\eta,\eta}(O(t-s)^\eta) \times L(s, P(s, z_2(s)))ds\|$$

therefore,

$$\|(Yz_1)(t) - (Yz_2)(t)\| \leq \int_0^t (t-s)^{\eta-1}$$
$$\|E_{\eta,\eta}(O(t-s)^\eta)\| \|L(s, P(s, z_1(s)))$$
$$- L(s, P(s, z_2(s)))\| ds$$

thus, by condition [a$_1$]

$$\|(Yz_1)(t) - (Yz_2)(t)\| \leq m_2 \left(\frac{h^\eta}{\eta}\right) m_3 m_4$$
$$\times \|z_1(t) - z_2(t)\|$$

Then condition [a$_3$] implies that

$$\|(Yz_1)(t) - (Yz_2)(t)\| \leq \|z_1(t) - z_2(t)\|.$$

Thus Y is nonexpansive mapping. Since $w(r)$ is closed, convex, bounded subset of (RBS) Q_n and $w(r)$ is weakly compact having normal structure (see example of Definition 3 and example 1), then by theorem 2, there exists a fixed point $z \in w(r)$ such that $(Yz)(t) = z(t)$ and hence this fixed point is a solution of Eq.4 on I, which satisfied $z(h) = z_1$, therefore the nonlinear control system Eq.4 is controllable on I. ∎

Now, from the obtained results in this work, the following important results can deduce.

Remark 2 : In this paper, the controllability of the solution $z(t) \in R^n$, $t \in [0, h]$ of the nonlinear control system Eq.4, where R^n is (RBS) and has Opial's condition (see Definition 2 with example) have been discussed. Thus, the previous results for control system Eq.4 when it is defined on arbitrary (RBS) having Opial's condition can extend by using the same manner in the proof of theorem 4.

Remark 3 : Let Q_n be only a Banach space with $0 \leq \lambda < 1$ in condition [a$_3$], then by using the same manner of theorem 4, the operator Y being a contraction mapping. Thus, by theorem 1 a unique fixed point which is a solution to the control system Eq.4 on I = $[0, h]$ is obtained.

Application:

In this section, theorem 4 is applied to obtain the result. Let $K \in L_1((o, h))$,

$$O = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0 < \eta < 1,$$

$t \in [0, h]$ and L is given as follows

$$L\left(t, z(t), \int_0^t K(t, s, z(s))ds\right) = \begin{pmatrix} 0 \\ \frac{\int_0^t e^{-z_1(s)}ds}{1+z_1^2(t)+z_2^2(t)} \end{pmatrix}$$

Now, consider the system

$$D^\eta z(t) =$$
$$Oz(t) + Bu(t) +$$
$$L\left(t, z(t), \int_0^t K(t, s, z(s))ds\right), z(0)=z_0 \quad (8)$$

Solution: Here $z(t) = (z_1(t), z_2(t))^*$
with $z_1(t) = z(t), D^{\frac{\eta}{2}} z_1(t) = z_2(t)$.

Let, $n_1 = \sum_{i=0}^\infty \frac{-1^i (h-s)^{2i\eta}}{\Gamma((2i+1)\eta)}$, $n_2 \sum_{i=0}^\infty \frac{(-1)^i (h-s)^{(2i+1)\eta}}{\Gamma(2\eta(i+1))}$.

Then Mittag-Leffler matrix function is

$$E_\eta(0t^\eta) =$$
$$\begin{pmatrix} \sum_{i=0}^\infty \frac{(-1)^i t^{2i\eta}}{\Gamma(2i\eta+1)} & \sum_{i=0}^\infty \frac{(-1)^i t^{(2i+1)\eta}}{\Gamma((2i+1)\eta+1)} \\ -\sum_{i=0}^\infty \frac{(-1)^i t^{(2i+1)\eta}}{\Gamma((2i+1)\eta+1)} & \sum_{i=0}^\infty \frac{(-1)^i t^{2i+\eta}}{\Gamma(2i\eta+1)} \end{pmatrix}$$

Also,

$$E_{\eta,\eta}(O(h-s)^\eta) =$$
$$\begin{pmatrix} \sum_{i=0}^\infty \frac{(-1)^i (h-s)^{2i\eta}}{\Gamma((2i+1)\eta)} & \sum_{i=0}^\infty \frac{(-1)^i (h-s)^{(2i+1)\eta}}{\Gamma(2\eta(i+1))} \\ -\sum_{i=0}^\infty \frac{(-1)^i (h-s)^{(2i+1)\eta}}{\Gamma(2\eta(i+1))} & \sum_{i=0}^\infty \frac{(-1)^i (h-s)^{2i\eta}}{\Gamma((2i+1)\eta)} \end{pmatrix}$$

It is easy to calculate the controllability matrix which defined in theorem 3

$$W = \int_0^h (h-s)^{\eta-1} \begin{pmatrix} n_2^2 & n_1 n_2 \\ n_1 n_2 & n_1^2 \end{pmatrix} ds$$

Thus W is a positive defined for any $h > 0$.
Therefore by theorem 3, the linear (FCS) Eq.1 is controllable on $[0, h]$.

Now, let $P = (t, z(t)) = \int_0^t K(t, s, z(s))ds$, then L satisfies the condition [a$_1$], and hence by theorem 4, the nonlinear control system Eq.8 is controllable.

Conclusion:

The controllability of (FCS) in arbitrary (RBS) by using fixed point theorem that deals with nonexpansive mapping is examined. For this purpose, then some preliminaries related to the solutions of (FCS) and to prove the main result which guarantees the sufficient condition for the controllability of considered system are given. An application is presented to illustrate the value of the obtained results.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
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Reference:

1. Balachandran K, Kokila J. Controllability of Nonlinear Implicit Fractional Dynamic Systems. IMA J Appl Math. 2014; 79: 562-570.

2. AL-Jawari, N J, Shaker S M. Controllability of Fractional Control Systems Using Schauder Fixed Point Theorem. AJBAS. 2016; 10(8):25-30.

3. Li Ding X, Nieto J. Controllability of Nonlinear Fractional Delay Dynamical Systems with Prescribed Controls. Nonlinear Analysis: Modeling and control. 2018; 23(1): 1-18.

4. Lizzy R M, Balachandran K. Boundary Controllability of Nonlinear Stochastic Fractional Systems in Hilbert Spaces. Int. J. Appl. Math. Comput. Sci. 2018; 28(1):123-133.

5. Mater M H. On Controllability of Linear and Nonlinear Fractional Integrodifferential System. FDC. 2019; 9(1): 19-32.

6. Nawaz M, Wei J, Sheng J, Niazi A, Yang L. On The Controllability of Nonlinear Fractional System with Control Delay. Hacet. J. Math. Stat. 2020; 49(1): 294-302.

7. Limaye B V. *Functional Analysis,* Second Edition, New Age International (p) Ltd., Publishers: New Delhi, Mumbai; 1996.

8. Denkowski Z, Migorski S, Papageorgiou N. An Introduction to Nonlinear Analysis: Applications. Kluwer Academic Publishers: NewYork, London; 2003.

9. Moosaei M. Fixed Points and Common Fixed Points for Fundamentally Nonexpansive Mappings on Banach spaces. J. hyperstructures. 2015;4(1): 50-56.

10. Dozo, E L. Multivalued Nonexpansive Mappings and Opial's Condition. Amer. Math. Soc. 1973; 38(2); 286-292.

11. Radhakrishnan, M, Rajesh S, Agrawal S. Some Fixed Point Theorem on Non-Convex Sets, Appl. Gen. Topol. 2017; 18(2): 377-390.

12. Kilbas A, Srivastava H, Trujillo J. Theory and Applications of Fractional Differential Equations. Elsevier: Amsterdam; 2006.

نتائج الوجود وإمكانية السيطرة لأنظمة السيطرة الكسرية في فضاءات بناخ الإنعكاسية باستخدام نظرية النقطة الصامدة

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الخلاصة :

تم عرض نظرية النقطة الصامدة للتطبيق غير الممدد وذلك لدراسة الوجود للحل والشروط الكافية لإمكانية السيطرة لأنظمة السيطرة الكسرية غير الخطية والمعرفة على فضاءات بناخ الانعكاسية. كذلك تم تعديل وتطوير النتيجة التي تم الحصول عليها اعلاه وذلك عندما يكون النظام معرف على اي فضاء يمتلك شرط Opial باستخدام نظرية النقطة الصامدة التي تتعامل مع التطبيق غير المدد والمعرف على مجموعة تمتلك البنية العمودية. ثم تم إعطاء تطبيق يوضح فعالية النتائج التي تم الحصول عليها.

الكلمات المفتاحية: امكانية السيطرة، نقطة صامدة، ، نظام سيطرة كسري، بنية عمودية ، شرط Opial.