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Total Global Dominator Coloring of Trees and Unicyclic Graphs

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Abstract:

A total global dominator coloring of a graph G is a proper vertex coloring of G with respect to which every vertex v in V dominates a color class, not containing v and does not dominate another color class. The minimum number of colors required in such a coloring of G is called the total global dominator chromatic number, denoted by $\chi_{tgd}(G)$. In this paper, the total global dominator chromatic number of trees and unicyclic graphs are explored.

Keywords: Global dominator coloring, Total domination number, Total dominator coloring, Total global domination number, Total global dominator coloring, Trees, Unicyclic graphs.

MSC2020: 05C15, 05C69

Introduction:

Let $G = (V, E)$ be a simple connected graph. An open neighbourhood of a vertex v, denoted by $N(u)$, is the set of all vertices $u \in V(G)$: $uv \in E(G)$ and a closed neighbourhood of v, denoted by $N[v]$, is $N(u) \cup \{v\}$. The open (closed) neighbourhood of a set D is the union of all open (closed) neighbourhood of all vertices in D. The open and closed neighbourhood of D is denoted by $N(D)$ and $N[D]$ respectively.

A set D is called a dominating set of G if $N[D] = V(G)$. If $uv \in E(G)$, then u is called a dominator of v or v is dominated by u and vice versa. If $N(D) = V(G)$, then D is called a total dominating set of G. The minimum cardinality of a dominating set of G is known as its domination number, denoted by $\gamma(G)$ and the minimum cardinality of a total dominating set of G is its total domination number, denoted by $\gamma_t(G)$. If D is a dominating set (total dominating set) of both G and complement of $\overline{G}$, then D is called a global dominating set (total global dominating set) of G and cardinality of a minimum global dominating set (resp. total global dominating set) is called the global domination number (resp. total global domination number) of G, denoted by $\gamma_g(G)$ $(resp. \gamma_{tg}(G))$ [1]. Many variations of domination have been widely studied [2,3].

The distance between two vertices u and v of a graph G, denoted by $d(u, v)$, is the number of edges in a minimal path between u and v. A tree is a connected acyclic graph. If two non-adjacent vertices in a tree are connected by an edge, then the new graph will consist of one (unique) cycle. Such graphs are known to be unicyclic graphs. In other words, unicyclic graphs are the graphs containing exactly one cycle in it.

A proper vertex coloring (or a proper coloring) of a graph G is a mapping from the vertex set of G to a non-empty set of colors such that adjacent vertices receive distinct colors. The chromatic number of G, denoted by $\chi(G)$ is the minimum number of colors in its proper coloring. Unless stated otherwise, all the vertex coloring discussed in this paper are proper.

Dominator Coloring of Graphs

Coloring and domination in graphs are two major areas of research in graph theory. Recently the concept dominator coloring, which is a combination of domination and coloring, has emerged as a promising area for further research.

A dominator coloring of G is a coloring such that every vertex in $V(G)$ dominates some color class and the minimum number of colors used to color G

in this way is called the dominator chromatic number of G, written by $\chi_d(G)$ [4]. A dominator coloring in which the open neighbourhood of every vertex contains at least one color class, is called a total dominator coloring (TD-coloring) and the minimum number of colors required for a TD-coloring is called the total dominator chromatic number of G, denoted by $\chi_d^t(G)$ [5]. Further studies on $\chi_d^t(G)$ and its variations can be found in [6,7].

Corresponding to a coloring of a graph G, a color class which is dominated by a vertex is known as a dom-color class whereas a color class such that no vertex of G is adjacent to any vertex in that color class is known as an anti-dom-color class of v. A coloring is called a global dominator coloring if each vertex $v \in V(G)$ has a dom color class and an antidom color class. Global dominator chromatic number of G is the minimum number of colors required for a global dominator coloring of G and is denoted by $\chi_{gd}(G)$ [8]. A color class is called a proper dom-color class of v if the $N(v)$ contains that color class. Using this definition, the notion of total global dominator coloring (tgd-coloring) has been introduced as a coloring for which every vertex has proper dom color class and an antidom color class. The minimum number of colors used in a tgd-coloring of G is called the total global dominator chromatic number of G and is denoted by $\chi_{tgd}(G)$ see [9]. Where the relationship between $\chi_{tgd}(G)$ with different graph parameters have been studied and χ_{tgd} of graphs such as paths, cycles, complete multipartite graphs, complement of paths and complement of cycles the For any graph G admitting tgd-coloring, $\chi_{tgd}(G) \geq 4$ and $1 \leq \delta(G) \leq \Delta(G) \leq n-2$.

This paper is an extension of the study of the parameter $\chi_{tgd}(G)$ by examining the tgd-coloring for the families of trees and unicyclic graphs. For the terminology and results of graph theory refer [10], for more about domination in graphs refer to [11] and for the terminology of graph coloring, see [12].

Some of the results which are significant and relevant in this study are listed below:

Theorem 1 [9] *For any graph G which admits tgd-coloring, $\chi_{tgd}(G) \leq \gamma_{tg}(G) + \chi(G)$.*

Observation 1 [6] *If T is a tree of order $n \geq 2$, then $\gamma_t(T) \leq \chi_d^t(T) \leq \gamma_t(T) + 2$.*

Main Results:

Let's begin with a general result that links the parameters $\chi_{tgd}(G)$ and $\chi_d^t(G)$ if $\delta(G) = 1$.

Theorem 2 *Let $\delta(G) = 1$. Then, $\chi_d^t(G) \leq \chi_{tgd}(G) \leq \chi_d^t(G) + 2$.*

Proof. If G admits a tgd-coloring, then $\chi_d^t(G) \leq \chi_{tgd}(G)$ [2]. It remains to prove that $\chi_{tgd}(G) \leq \chi_d^t(G) + 2$.

Consider a χ_d^t-coloring c of G such that $\chi_d^t(G) = k$. From the very definition of TD-coloring it follows that every vertex of G has a proper dom-color class. Since $\delta(G) = 1$, there exists a support vertex, say v in G. Let w be a pendant vertex adjacent to v and u be a non-neighbour of v in G. Now define a new coloring c' such that $c'(v_i) = c(v_i)$ for all $v_i \in V(G)$, except for the vertices u and w and assign two new colors to the vertices u and w. Then it is easy to verify tht c' is a tgd-coloring of G. Every vertex of G has a proper dom-color class with respect to the coloring c' as c is a χ_d^t-coloring. Further w being a pendant vertex having a unique color, $\{w\}$ acts as an anti dom-color class for all vertices of G other than the vertices v, w. Again $\{u\}$ acts as an anti-dom-color class of vertices v and w. Hence c' is a tgd-coloring of G, proving that $\chi_{tgd}(G) \leq \chi_d^t(G) + 2$. □

The bounds in the inequality of the above theorem are sharp. The examples for both cases are in the coming sections.

Let s be the number of support vertices of G. Then, the following theorem establishes a lower bound for $\chi_{tgd}(G)$ in terms of s.

Theorem 3 *If G admits tgd-coloring, then $\chi_{tgd}(G) \geq s + 1$.*

Proof. The only possible member of a proper dom-color class of a pendent vertex is its support vertex. Therefore, the cardinality of the color class of a support vertex is 1. Since there are s support vertices and cardinality of each of the support vertices is 1, $s + 1$ colors are needed to color the all the support vertices and at least one more color is need to color the remaining vertices of G. That is, $\chi_{tgd}(G) \geq s + 1$. □

The bound of the above theorem is sharp. It can be verified with the help of an example. Consider the graph given in Fig. 1, which has four support vertices. That is, $s = 4$. If we color the four support vertices with 4 different colors and all the remaining vertices with a color different from the four colors used, then the defined coloring is a tgd-coloring. Therefore, $\chi_{tgd}(G) \leq 5$. By Theorem 3, $\chi_{tgd}(G) \geq s + 1 = 5$. From the above two inequalities, $\chi_{tgd}(G) = 5 = s + 1$. □

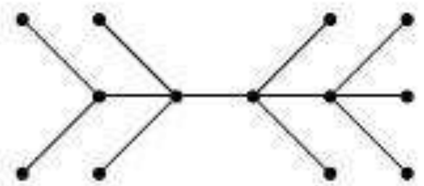

Figure 1. A graph G with $\chi_{tgd}(G) = 5 = s + 1$.

Theorem 4 *If G is a graph having at least two support vertices such that the distance between them is at least 3, then $\chi_d^t(G) = \chi_{tgd}(G)$.*
Proof. Assume that G admits a tgd-coloring such that G contains at least two support vertices which are at a distance of more than two. Obviously G admits a χ_d^t coloring. Moreover, under any TD-coloring of G, the colors received by the support vertices would be unique. Since there exist at least two support vertices which are at a distance more than 2, the closed neighbourhood of these two vertices will not have a common vertex. Therefore, one of these support vertices will constitute the anti-dom-color class for any vertex of G. Hence any χ_d^t coloring of G will also be a tgd coloring. □
The converse of the above theorem is not true. For example, consider the path P_5. Here, $\chi_d^t(P_5) = \chi_{tgd}(P_5) = 4$, even though there are no two support vertices which are at a distance at least 3.

Trees

This section discusses the condition for a tree T to admit tgd-coloring and determine the values of χ_{tgd} of trees in terms of the graph parameters such as χ_d^t, γ_{tg}, the diameter $diam(T)$, and the number of support vertices s.
The following result presents a necessary and sufficient condition for any tree to admit a tgd-coloring.

Theorem 5 *A tree T admits a tgd-coloring if and only if $diam(T) \geq 3$.*

Proof. Let T admits a tgd-coloring. If possible, assume that $diam(T) < 3$. In this case the central vertex of T is adjacent to all other vertices. That means the central vertex does not have an anti-dom-color class, contradicting the assumption that T admits a tgd-coloring. Therefore, $diam(T) \geq 3$.

Conversely assume that T is a tree such that $diam(T) \geq 3$. Assign distinct colors to all vertices of T, which obviously gives rise to a proper dom-color class to each vertex of T. Since the $diam(T) \geq 3$, corresponding to any vertex of T there exist at least one non-adjacent vertex serving as the anti-dom-color class. □

In view of Theorem 5, all trees considered for further discussion would be trees of diameter at least 3. The following theorem gives the exact value of χ_{tgd} for trees with diameter 3.

Theorem 6 *If $diam(T) = 3$, then $\chi_{tgd}(T) = 4$.*
Proof. Consider a tree T of diameter 3. Then by Theorem 5, T admits a tgd-coloring and hence $\chi_{tgd}(G) \geq 4$. Now it remains to show the existance of tgd- coloring that requires only four colors. T being a tree with diameter 3 contains exactly two support vertices. Use two distinct colors to color these support vertices. Color the pendant vertices adjacent to one of the support vertices with color 3 and the remaining vertices with color 4, is a tgd-coloring, thus completing the proof. □
Next theorem provides χ_{tgd} of a tree with diameter 4 in terms of its number of support vertices.

Theorem 7 *If $diam(T) = 4$, then $\chi_{tgd}(T) = s + 2$*
.
Proof. Let $diam(T) = 4$. Note that trees of diameter 4 can be classified into two types as given in Fig. 2.

Type– 1 Type -2

Figure 2. Two types of trees with diameter 4.

The first being trees for which the central vertex is not a support vertex. This class of trees may be referred to as Type 1. The second type are those trees whose central vertex is a support vertex as well. The theorem can be proved by considering these two cases separately.

Case 1: Let T be a tree of Type-1.
In this case the center of T, say u, is not a support vertex. Then by Theorem 2, $\chi_{tgd}(T) \geq s + 1$.
It can be claimed that $\chi_{tgd}(T) \neq s + 1$. On the contrary, assume that $\chi_{tgd}(T) = s + 1$. Then, the colors assigned to the support vertices should be distinct. Hence there exist only one color that remains to color the pendent vertices and the central vertex u which is not possible as this coloring does not yield an anti-dom-color class for u. Therefore $\chi_{tgd}(T) \geq s + 2$. Now it remains to show that T admits a tgd-coloring with $s + 2$ colors. Define a coloring scheme such that the s support vertices are colored with s different colors and the central vertex

u with a color different from the s colors used and the remaining vertices with yet another new color. This yields a tgd-coloring and hence $\chi_{tgd}(T) = s + 2$.

Case 2: Let the tree is of Type-2.

In this case, the center of T, say u, is a support vertex. By Theorem 2, $\chi_{tgd}(T) \geq s + 1$, first show that $\chi_{tgd}(T) \neq s + 1$. If possible, assume that T is a tree with $\chi_{tgd}(T) = s + 1$. Since s colors required to color the support vertices, the remaing all vertices should be color with the remaining one color. But it is not possible since in this case u would not have an anti-dom-color class, implies $\chi_{tgd}(T) \geq s + 2$. To prove $\chi_{tgd}(T) = s + 2$ it is enough to show that there exists a tgd-coloring of T with $s + 2$ colors. Define a coloring of T as follows. Color all the s support vertices with s different colors, the support vertices at u with $s + 1$-th color and the remaining vertices with $s + 2$-th color, yields a tgd-coloring. □

Remark 1 The result of Theorem 7 is true for trees of diameter 3 as well.

The following theorem establishes the relationship between χ_{tgd} and χ_d^t for trees with diameter 4.

Theorem 8 *Let T be a tree with diameter 4. Then,*

i) $\chi_{tgd}(T) = \chi_d^t(T)$, if the center of T is not a support vertex.

ii) $\chi_{tgd}(T) = \chi_d^t(T) + 1$, if the center of T is a support vertex.

Proof. This follows from Theorem 7 and the result that, $\chi_d^t(T) = s + 2$ for trees of Type-1 and, $\chi_d^t(T) = s + 1$ for trees of Type-2 [5]. □

In view of the above theorem note that, if, the set of support vertices of a tree of diameter 4 form an independent set, then $\chi_d^t(T) = \chi_{tgd}(T)$.

Theorem 9 *For a tree T with diameter more than 4, $\chi_{tgd}(T) = \chi_d^t(T)$.*

Proof. Let T be a tree with diameter more than 4, T contains at least two support vertices such that the distance between them is 3. Therefore, the proof follows from Theorem 3. □

In view of the Theorem 9 note that, if, the set of support vertices of a tree of diameter 4 form an independent set, then $\chi_d^t(T) = \chi_{tgd}(T)$. And from Theorem 2 it obvious that $\chi_d^t(T) \leq \chi_{tgd}(T) \leq \chi_d^t(T) + 2$. □

Next theorem is a characterization of trees.

Theorem 10 *For a tree $\chi_{tgd}(T) = \chi_d^t(T) + 1$ if and only if T is of diameter 3 or of diameter 4 and the center vertex is a support vertex.*

Proof. By Theorem 9, all trees of diameter 5 or above $\chi_{tgd}(T) = \chi_d^t(T)$. Hence it remains to consider only trees of diameter 3 or 4. For a tree T of diameter 3, one may get $\chi_{tgd}(T) = 4 = \chi_d^t(T) + 1$. Also, if T is of diameter 4 then by Theorem 7, $\chi_{tgd}(T) = \chi_d^t(T)$ if center of T is not a support vertex. Therefore, $\chi_{tgd}(T) = \chi_d^t(T) + 1$ if and only if T is of diameter 3 or of diameter 4 and the center vertex is a support vertex. □

In view of the results mentioned above, the following inequality is immediate. All trees which admit tgd-coloring satisfies Theorem 2. From the above results it is obvious that the maximum value of $\chi_{tgd}(T)$ is $\chi_d^t(T) + 1$. Therefore, one has the following observation.

Observation 2 *For a tree T with diameter at least 3, $\chi_d^t(T) \leq \chi_{tgd}(T) \leq \chi_d^t(T) + 1$.*

For trees which admit tgd-coloring, the next theorem is a characterization with respect to its support vertices.

Theorem 11 *For a tree T, $\chi_{tgd}(T) = s + 1$ if and only if the following conditions hold.*

i) $diam(T) \geq 5$;

ii) If u is a support vertex of T then there exist another support vertex v of T such that $d(u, v) = 1$; and

iii) The minimum distance between any two non-support non-leaf vertices is 2.

Proof. Let T be a tree with the given three conditions. Consider a proper coloring c such that all the support vertices in T receive distinct colors, say $c_1, c_2, \dots, c_s$ and all the remaining vertices receive same color, say c_{s+1}, which is different from all $c_i; 1 \leq i \leq s$. This coloring is possible, as the distance between two non-support vertices is at least 2. Since $\chi_{tgd}(T) \geq s + 1$, it is enough to prove that c is a tgd-coloring. Let v be an arbitrary vertex of T. Then, there exists a support vertex v' in T that is adjacent to v. That is, $\{v'\}$ is a proper dom-color class of the vertex v. since $diam(T) \geq 5$, there exist two support vertices u and u' in T such that $d(u, u') \geq 3$. Therefore, either $\{u\}$ or $\{u'\}$ is an anti-dom color class of v. Therefore, c is a tgd-coloring.

Conversely, assume that $\chi_{tgd}(T) = s + 1$. The first condition is immediate from Theorem 6 and Theorem 7. Let u be a support vertex of T which is not adjacent to any other support vertex. Therefore, the only possible proper dom-color class of u is the color class of the leaf vertices. Since $diam(T) \geq 5$, there is at least one leaf which is not adjacent to u. This leads to a contradiction that u does not have a proper dom-color class. This proves the second condition. If there exist two adjacent non-leaf non-support vertices, then $\chi_{tgd}(T) = s + 1$ as these two vertices cannot have the same color. Thus, the third condition also holds. □

A caterpillar graph is a tree where the removal of its pendent vertices gives a path of order greater than or equal to 2. A complete caterpillar graph T is a caterpillar where $V(T)$ contains only support vertices and pendant vertices. The following result follows from Theorem 11.

Remark 2 *For any complete caterpillar graph T of diameter at least 5, $\chi_{tgd}(T) = s + 1$.*

The following theorem establishes a relationship between χ_{tgd} and γ_{tg} of a tree.

Theorem 12 *For any tree T of order greater than or equal to 4, $\chi_{tgd}(T) \leq \gamma_{tg}(T) + 2$.*

Proof. Let T be a tree of order greater than 3, $\chi(T) = 2$. Therefore, by Theorem 1, $\chi_{tgd}(T) \leq \gamma_{tg}(T) + 2$. □

Theorem 13 *Let a tree T admits tgd-coloring. Then, $\gamma_t(T) \leq \chi_{tgd}(T) \leq \gamma_t(T) + 2$.*

Proof. Assume *tree T admits tgd-coloring* Let $diam(T) \geq 5$. Then $\chi_{tgd}(T) = \chi_d^t(T)$. Therefore, for trees with diameter grater than or equal to 5 the result follows from Observation 1. Hence, need to consider only trees of diameter 3 and 4.

If T is of diameter 3, $\gamma_t(T) = 2$ and $\chi_{tgd}(T) = 4$. That is, $\chi_{tgd}(T) = \gamma_t(T) + 2$. Next, consider trees of diameter 4. Then there are two cases. If T is a tree of Type-1, then it is seen that $\gamma_t(T) = s + 1$. By Theorem 7 $\chi_{tgd}(T) = s + 2$. Therefore, $\chi_{tgd}(T) = \gamma_t(T) + 1$. If T is a tree of Type-2, then $\gamma_t(T) = s$. By Theorem 7, $\chi_{tgd}(T) = s + 2$. Therefore, $\chi_{tgd}(T) = \gamma_t(T) + 2$. Therefore, $\gamma_t(T) \leq \chi_{tgd}(T) \leq \gamma_t(T) + 2$. □

From the above proof, examples for trees with $\chi_{tgd}(T) \leq \gamma_t(T) + 1$ and $\chi_{tgd}(T) \leq \gamma_t(T) + 2$ are obtained. P_6 is an example for a tree where $\chi_{tgd}(T) \leq \gamma_t(T)$.

In [6], Henning characterized trees for which $\chi_d^t(T) = \gamma_t(T)$, $\chi_d^t(T) = \gamma_t(T) + 1$ and $\chi_d^t(T) = \gamma_t(T) + 2$. As for all trees with diameter greater than 4, $\chi_{tgd}(T) = \chi_d^t(T)$, one can characterize all trees of diameter greater than 4 for which $\chi_{tgd}(T) = \gamma_t(T)$, $\chi_{tgd}(T) = \gamma_t(T) + 1$ and $\chi_{tgd}(T) = \gamma_t(T) + 2$ by replacing $\chi_d^t(T)$ by $\chi_{tgd}(T)$ in the results corresponding in [6]. For trees of diameter 3 and 4 can seen in Theorem 13.

Unicyclic Graphs

In this section, let C denote the cycle in the unicyclic graph G. If G is a unicyclic graph, a vertex v lies on C is called an extreme vertex if it is adjacent to a vertex of degree at least 2 lying outside C. It is known that the minimum value of χ_d^t and χ_{tgd} for a graph G are 2 and 4 respectively. In this section, the total dominator chromatic number and total global dominator chromatic number of unicyclic graphs are studied.

Theorem 14 *If G is a unicyclic graph other than C_4, then $\chi_d^t(G) \geq 3$. Further, $\chi_d^t(G) = 3$ if and only if G is isomorphic to one of the graphs in Fig. 3.*

Proof. Note that the total dominator chromatic number of any cycle other than C_4 is more than 2. Therefore, the proof of the first part is trivial. Next, it is to be proved the necessary and sufficient condition for $\chi_d^t(G) = 3$. It is easy to verify that χ_d^t of each of the graphs in Fig. 3 is 3. So, it remains to prove that $\chi_d^t(G) \neq 3$, for any unicyclic graph which is not present in Fig. 3. It is known that χ_d^t of any cycle of length other than 3 or 4 is greater than 3. Therefore, one must consider unicyclic graphs with cycle of length 3 and 4.

Case 1: Let G be a unicyclic graph which has a cycle C of length 3. In this case, there exist two types of unicyclic graphs, one without extreme vertices and the other is with extreme vertices. First, let us assume that G has an extreme vertex. Therefore, there exists a support vertex outside C. Since the color of a support vertex cannot be assigned to any other vertex in TD-coloring, another three colors are required to color the vertices of the cycle. In this case, $\chi_d^t(G) \geq 4$. If G does not have an extreme vertex, then either G is a cycle C, or the graph obtained by adding pendent vertices at the vertices of C. Here, the number of vertices of C at which pendent vertices can be added such that $\chi_d^t(G) = 3$ is to be determined. It can be observed that if pendent vertices are added at one or two vertices of C, then $\chi_d^t(G) = 3$. But, if pendent vertices are added at all three vertices of C, then $\chi_d^t(G) = 5$. Therefore, the unicyclic graphs which have a cycle of length 3 with $\chi_d^t(G) = 3$ are those present in Fig. 3.

Case 2: Let G be a unicyclic graph which has a cycle C of length 4. Also, let $\chi_d^t(G) = 3$. Therefore, G should not have an extreme vertex. Because if G contains an extreme vertex there exists a support vertex outside the cycle say u, whose color class is solitary. If the remaining vertices of G are colored with another two colors, then u will not have a proper dom-color class. Next, it is required to find which of

C can be a support vertex. If pendent vertices are added only at a vertex of C, then $\chi_d^t(G) = 3$. Since $\chi_d^t(G) = 3$, it is not possible to add pendent vertices at any three vertices of C. Suppose that support vertices are added at two adjacent vertices, at least two more colors are required to color G, which leads to a contradiction. Let the non-adjacent vertices of C be support vertices. If the two support vertices are colored using two distinct colors and the remaining vertices with a third color, then the support vertices will not have a proper dom color class. That is, $\chi_d^t(G) \geq 3$, which is a contradiction. Therefore, a unicyclic graphs with a cycle C of length 4 with $\chi_d^t(G) = 3$ is as shown in Fig. 3. □

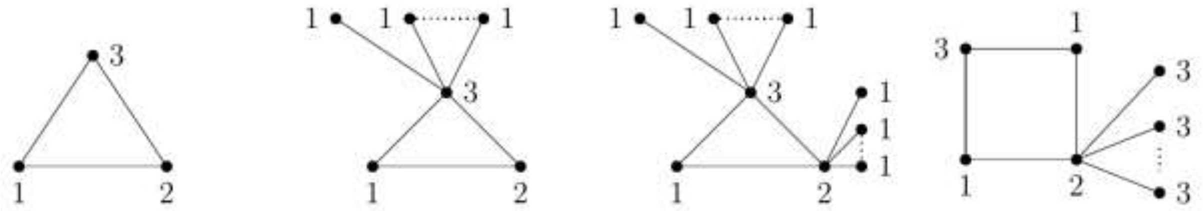

Figure 3. Unicyclic graphs with $\chi_d^t = 3$

Theorem 15 *Let G be a unicyclic graph and C be the cycle in G. Then under any TD-coloring of G, every vertex not on C has an anti dom-color class.*

Proof. Let the vertices of the cycle C in G be $v_1, v_2, v_3, \ldots, v_k, v_1$. If G does not contain an extreme vertex , that is, the vertices outside C are pendant vertices. Consider a TD- coloring of G. The color of a support vertex is solitary. In any proper coloring of C, at least two colors are present. Therefore, at least two more colors are used in the TD- coloring of C and one of them act as an anti dom-color class of the pendant vertex. That is, all pendant vertices of G have an anti dom-color class. If G contains an extreme vertex, say v_1. Let T be a branch at v_1 and consider w be a support vertex in T. Then, the color class $\{w\}$ act as an anti dom-color class of every vertex of $G - C$ not on T. For vertices in $T - \{v_1\}$, color class of v_3 or proper dom-color class of v_3 act as an anti dom-color class. This completes the proof. □

Consider unicyclic graphs which are not cycles, Theorem 2 hold. Also, C_n is tgd-colorable if $n \geq 4$ and if $n \geq 5, \chi_{tgd}(G) = \chi_d^t(G)$. For C_4, $\chi_{tgd}(G) = \chi_d^t(G) + 2$.

Theorem 16 *Let G be a unicyclic graph that admits tgd-coloring. Then, $\chi_d^t(G) \leq \chi_{tgd}(G) \leq \chi_d^t(G) + 2$.*

Proof. Let G be a unicyclic graph that admits tgd-coloring. If G is a unicyclic graph with $\delta(G) = 1$ then by Theorem 2, $\chi_d^t(G) \leq \chi_{tgd}(G) \leq \chi_d^t(G) + 2$. It remains to prove that cycles also admit this bound. It is proved in [9] that $\chi_{tgd}(G) = \chi_d^t(G)$, where G is a cycle of length at least 5 and for cycle of length 4, $\chi_{tgd}(G) = \chi_d^t(G) + 2$. □

Theorem 17 *If G is a unicyclic graph with a cycle of length at least 5, then, $\chi_d^t(G) \geq 4$. Further, $\chi_d^t(G) = 4$, then the length of the cycle is 5 or 6.*

Proof. As G contains a cycle C of length more than 4, the first part of the theorem is clear. Assume that $\chi_d^t(G) = 4$. It remains to prove that the length of the cycle is 5 or 6. Let the length of the cycle be at least 7. Note that $\chi_d^t(C) \geq 5$ for cycle C of length at least 7. Therefore, $\chi_d^t(G) > 4$ for unicyclic graphs with cycle of length more than 6. That is, the length of the cycle is either 5 or 6. □

Theorem 18 *Let G be a unicyclic graph with two or more extreme vertices. Then, $\chi_{tgd}(G) = \chi_d^t(G)$.*

Proof. Let G be a unicyclic graph with at least two extreme vertices. Then, there exist at least two support vertices such that the distance between them is at least three. Therefore, by Theorem 4, $\chi_d^t(G) = \chi_{tgd}(G)$. □

Theorem 19 *If G is a unicyclic graph without extreme vertices and $\chi_d^t(G) \geq 5$, then $\chi_{tgd}(G) = \chi_d^t(G)$.*

Proof. Let G be a unicyclic graph with no extreme vertex. Also, let $\chi_d^t(G) \geq 5$ with respect to the coloring C. To prove C is a tgd-coloring, it is required to show that all vertices of G has an anti dom-color class. One may note that each support is solitary. Let v be an arbitrary vertex of G. If v is not a support vertex, then $deg(v) \leq 2$. As $\chi_d^t(G) \geq 5$, v is not adjacent to any of the vertices in at least 2 color classes. That is, v has an anti dom-color class. If v is a support vertex of G, then v lies on C and v can be adjacent to vertices in at most three color classes. Therefore, there exist at least one color class which contains no vertex adjacent to v. That is, v has an anti dom-color class with respect to C. Therefore, $\chi_{tgd}(G) = \chi_d^t(G)$. □

Conclusion:

In this article, some general results on the total global dominator coloring of graphs have been reported and following to these, the total global dominator coloring of unicyclic graphs have also been investigated. This area is much promising for further research. Some of the open problems that were identified during the study are given below:

1. Characterize trees for which
   (a) $\chi_{tgd}(T) = \gamma_{tg}(T)$.
   (b) $\chi_{tgd}(T) = \gamma_{tg}(T) + 1$.
   (c) $\chi_{tgd}(T) = \gamma_{tg}(T) + 2$.

2. Find an efficient algorithm to determine χ_{tgd} of trees.

Authors' declaration:

- Conflicts of interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re- publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in CHRIST (Deemed to be University), India.

Authors' contributions statement:

The first author proposed and presented the idea. The first author developed the paper through continuous discussions with the second author. The second author proposed corrections and modifications of proofs and presentation style of the draft paper. The final copy was prepared through discussion by both the authors.

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إجمالي تلوين المسيطر العالمي للأشجار والرسوم البيانية أحادية الحلقة

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الخلاصة:

إن التلوين الشامل للمسيطر العام للرسم البياني G هو تلوين رأس مناسب لـ G فيما يتعلق بكل رأس v في V يهيمن على فئة اللون، لا تحتوي على v ولا تهيمن على فئة لون أخرى. يُطلق على الحد الأدنى لعدد الألوان المطلوبة في مثل هذا التلوين لـ G إجمالي عدد لوني للمسيطر العالمي، يُشار إليه بـ (χ_tgd (G. في هذا البحث، تم استكشاف العدد اللوني الشامل المسيطر للأشجار والرسوم البيانية أحادية الحلقة.

الكلمات المفتاحية: تلوين المسيطر العالمي، العدد الإجمالي للسيطرة، إجمالي التلوين المسيطر، العدد الإجمالي للسيطرة العالمية، إجمالي تلوين المسيطر العالمي، الأشجار، الرسوم البيانية أحادية الحلقة.