Some Properties of the Nonoscillatory Solutions of Second Order Linear Neutral Differential Equations

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Abstract: In this Paper the second order neutral differential equations are investigated, were we give some new sufficient conditions for all nonoscillatory solutions of equation (1.1) to converge to zero or to diverge.

1. Introduction

Consider the neutral differential equation

$$(1,1)\qquad [x(t)+p(t)x(\tau(t))]'' + q(t)x(\sigma(t)) = 0, \quad t \ge t_0$$

Under the standing hypotheses:

(1) $p \in C[[t_0,\infty),(0,\infty)]$;

(2) $\tau, \sigma \in C[[t_0,\infty), R]$, σ, τ are strictly increasing and

$\lim_{t\to\infty} \tau(t) = \infty$ *and* $\lim_{t\to\infty} \sigma(t) = \infty$.

(3) $q \in C[[t_0,\infty), R]$, q is continuous and $q(t)$ not equivalent to zero.

Our aim is to obtain new sufficient conditions for nonoscillatory solution of equation (1.1) to converge to zero or diverges.

By a solution of equation (1.1) we means a continuous function $x:[t_x,\infty)\to R$ such that $x(t)+p(t)x(\tau(t))$ is two times continuously differentiable, and $x(t)$ satisfy equation (1.1) for all sufficiently large $t \ge t_x$. A solution of (1.1) is said to be oscillatory if it has infinite sequence of zeros tending to infinity, otherwise is said to be nonoscillatory.

The problem of oscillation and nonoscillation for neutral differential equations has received considerable attention in recent years, see e.g.[1-8] and the references cited therein. However some results in this paper are new and the other ones in many cases complete the previous ones.

2. Some basic lemmas

The following lemmas will be used in the proof of the main results

Lemma 2.1. Let $f \in C^n(R,R)$ and

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$f^{(n)} f^{(n-1)} > 0$,then the following statements are satisfied

1. If $f^{(n)}$ is positive , then $f^{(i)}$ is increasing and
$$\lim_{t\to\infty} f^{(i)}(t)=\infty, \qquad i=n-1,n-2,\ldots,0$$
2. If $f^{(n)}$ is negative then $f^{(i)}$ is decreasing and
$$\lim_{t\to\infty} f^{(i)}(t)=-\infty, \qquad i=n-1,n-2,\ldots,0$$

Proof. *See* [6].

Lemma ***2.2*** Let $p(t)\in C([t_0,\infty);(0,\infty))$ such that $0 \le p_1 \le p(t) \le p_2$, and $q(t)\le q<0$, let $x(t)$ be an eventually positive solution of equation (1.1), and let $u(t)=x(t)+p(t)x(\tau(t))$, for $t\ge t_0$. Then the following statements are true:

(a) The function $u(t)$ and $u'(t)$ are strictly monotone and either

(2.1) $\lim_{t\to\infty} u(t)=\infty, \quad \lim_{t\to\infty} u'(t)=\infty$

or

(2.2) $\lim_{t\to\infty} u(t)=0, \quad \lim_{t\to\infty} u'(t)=0$

$u(t)>0\,, \quad u'(t)<0$

(b) Assume that $x(t)$ be bounded, then (2.2) holds, in particular $u(t)$ is bounded

Proof. See [8].

3. *Asymptotic behavior of equation (1.1)*

In this section we investigate the converges and the diverges of this solution of equation (1.1), define the function $u(t)=x(t)+p(t)x(\tau(t))$, then equation (1.1) will be

(3.1) $u''(t)=-q(t)x(\sigma(t))$

Theorem 3.1. Suppose that

$p(t)\ge 0\,, \quad q(t)\ge 0\,, \quad \sigma(t)>t\,, \quad t\ge t_0$ *and*

(3.2) $\int^{\infty} q(s)\,ds$

Then every bounded solution of equation (1.1) is either oscillatory or nonoscillatory such that $\liminf_{t\to\infty} |x(t)|=0$

Proof. Assume that *x(t)* be bounded and an eventually positive solution of (1.1). Then $u''(t)\le 0$ for all large *t*. We have only the case:

$u'(t)>0\,, \quad u(t)>0 \quad for \quad t\ge t_1\ge t_0$

to discuss.

Integrating equation (3.1) from t_1 *to* t we obtain

$$u'(t)-u'(t_1)=-\int_{t_1}^{t} q(s)x(\sigma(s))\,ds$$

as $t\to\infty$, we get

$$l-u'(t_1)=-\int_{t_1}^{\infty} q(s)x(\sigma(s))\,ds$$

(3.3) $$u'(t_1)=\int_{t_1}^{\infty} q(s)x(\sigma(s))\,ds+l$$

thus , if $\liminf_{t\to\infty} x(t)\ne 0$, then this implies that $\liminf_{t\to\infty} x(t)=c_1$, and

$x(t)>\frac{c_1}{2}>0$ hence,

$x(\sigma(t))>\frac{c_1}{2}$ for $t\ge t_2\ge t_1$

which gives a contradiction with (3.3).

Theorem 3.2. Suppose that $p(t)\ge 0$ is bounded $q(t)\le 0\,, \quad \sigma(t)>t\,, \quad for \quad t\ge t_0$ and

(3.4) $\int^{\infty} |q(s)|\,ds=\infty$

Then every bounded solution of equation (1.1) is either oscillatory or nonoscillatory such that $\liminf_{t\to\infty} |x(t)|=0$

Proof. The proof is similar to that in theorem 3.1 and we omitted it.

Example 3.1. Consider the neutral delay differential equation:

$$\frac{d^2}{dt^2}[x(t)+\frac{2}{t}x(2t)]-(1+3t^{-t})x(\frac{t^2}{2})=0, \quad t>1$$

all conditions of theorem 3.2 are satisfied , then all bounded solutions of the above equation are either oscillatory or nonoscillatory with $\liminf_{t\to\infty}|x(t)| = 0$, for instance $x(t) = \frac{c}{t}$ is such solution ($c \neq 0$) is constant.

Theorem 3.3. Suppose, that

$1 < \lambda \leq p(t) \leq \beta, \quad q(t) \leq 0, \quad \tau(t) < t,$
$\tau(t) < \sigma(t), \quad t \geq t_0$

and *(3.2)* holds. Then all bounded solutions of equation (1.1) are either oscillatory or nonoscillatory tends to zero as $t \to \infty$.

Proof. Assume that $x(t)$ be nonoscillatory bounded solution of equation(1.1),without loss of generality assume that $x(t)$ be an eventually positive. Then from equation (3.1) we get $u''(t) \geq 0$ for all large t. We claim that $u'(t) < 0$. otherwise $u'(t) > 0$ and by lemma 2.1 $u(t)$ is unbounded, which is a contradiction since $u(t)$ is bounded. So we have only the case

$$u''(t) \geq 0, \quad u'(t) < 0, \quad u(t) > 0$$

for $t \geq t_1 \geq t_0$ to show $\lim_{t\to\infty} x(t) = 0$, we have to show $\limsup_{t\to\infty} x(\tau(t)) = 0$ otherwise,

$$\limsup_{t\to\infty} x(\tau(t)) = c > 0$$

we claim that $\liminf_{t\to\infty} x(\tau(t)) = 0$

otherwise $\liminf_{t\to\infty} x(\tau(t)) = c_1$

which means that $x(\tau(t)) > \frac{c_1}{2}, \quad x(\sigma(t)) > \frac{c_1}{2}$ for $t \geq t_2 \geq t_1$.

Form (3.1) we get

$$u'(t) - u'(t_2) = -\int_{t_1}^{t} q(s)x(\sigma(s))\,ds$$

which as $t \to \infty$, leads to a contradiction since $\lim_{t\to\infty} u'(t) = \infty$, but $u'(t) < 0$ then clearly

$\liminf_{t\to\infty} x(\tau(t)) = 0$ and

$\limsup_{t\to\infty} x(\tau(t)) = c$, so there exist two sequences $\{t_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty}$ such that, $\liminf_{t\to\infty} x(\tau(t_n)) = 0$ and

$\limsup_{t\to\infty} x(\tau(t_n)) = c$ so, for any $\epsilon > 0$ there is $N \in \mathrm{N}$ such that $x(\tau(t_n)) < \epsilon$, $x(\tau(s_n)) > c - \epsilon$ for each $n \geq N$ and we can choose N such that $x(t_n) < c + \epsilon$ for $n \geq N$. Now if we choose the positive integers i, j such that $s_j > t_i$ and estimate the deference.

$$u(s_j) - u(t_i) = x(s_j) + p(s_j)x(\tau(s_j)) - x(t_i) + p(t_i)x(\tau(t_i))$$

$$> p(s_j)x(\tau(s_j)) - x(t_i) + p(t_i)x(\tau(t_i))$$

$$> \lambda(c - \epsilon) - c - \epsilon - \beta\epsilon$$
$$= (\lambda - 1)c - \epsilon(1 + \lambda + \beta)$$

We can choose ϵ small enough such that

$$(\lambda - 1)c - \epsilon(1 + \lambda + \beta) > 0$$

so that

$$u(s_j) > u(t_i)$$

where $s_j > t_i$ which is a contradiction, since $u(t)$ is decreasing. Hence

$$\limsup_{t\to\infty} x(\tau(t)) = 0$$

which is implies $\lim_{t\to\infty} x(t) = 0$.

Theorem 3.3. Suppose that

$0 < \lambda_1 \leq p(t) \leq \lambda_2, \quad q(t) \leq q < 0 \quad for \quad t \geq t_0.$

Then all nonoscillatory solutions of equation (1.1) are either tend to zero or

$|x(t)| \to \infty \quad as \quad t \to \infty.$

Proof. Assume that x (t) be nonoscillatory solution of equation (1.1), and suppose that x(t) be an eventually positive, and there exists t_0 such that $x(\tau(t)) > 0$ for to rove that either $\lim_{t\to\infty} x(t) = \infty$ *or* $\lim_{t\to\infty} x(t) = 0$.

Suppose that $\lim_{t\to\infty} x(t) \neq \infty$.

Then there exists $t_1 \geq t_0$ such that $x(t)$ is bounded for $t \geq t_1$, and there exists c such that

$$\limsup_{t\to\infty} x(t) = c \geq 0$$

We claim that $c = 0$ otherwise, $c > 0$. Then by Lemma 2.2-a we have two cases:

Case 1.

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = \infty$$

Which means $u(t)$ is unbounded, which is a contradiction, since $x(t)$ and $p(t)$ are bounded.

Case 2.

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0$$

and we have

$$\lim_{t\to\infty}\sup x(t) = c > 0$$

Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that, $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} x(t_n) = c$, and $\lim_{n\to\infty} u(t_n) = 0$ then

$$u(t_n) = x(t_n) + p(t_n)x(\tau(t_n)) \geq x(t_n) + \lambda_1\, x(\tau(t_n)),$$

as $n \to \infty$ we get

$$\lim_{n\to\infty} x(\tau(t_n)) \leq \frac{\lim_{n\to\infty} u(t_n) - \lim_{n\to\infty} x(t_n)}{\lambda_1}$$

$$= \frac{-c}{\lambda_1} < 0,$$

Which is a contradiction. Then either

$$\lim_{t\to\infty} x(t) = \infty, \quad or \quad \lim_{t\to\infty}\sup x(t) = 0$$

, which implies that $\lim_{t\to\infty} x(t) = 0$.

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بعض خواص الحلول غير المتذبذبة للمعادلات التفاضلية المحايدة من الرتبة الثانية

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الخلاصة

لاحظ المعادلة التفاضلية المحايدة من الرتبة الثانية

$$\left[x(t)+p(t)x(\tau(t))\right]'' + q(t)x(\sigma(t)) = 0, \qquad t \ge t_0$$

عندما $p \in C[[t_0,\infty),(0,\infty)]; \quad \tau,\sigma \in C[[t_0,\infty),R],$

وأن σ,τ هما متزايدة ، وأن $\lim_{t\to\infty}\sigma(t)=\infty, \quad \lim_{t\to\infty}\tau(t)=\infty$

$q \in C[[t_0,\infty),R]$، مستمرة وأن $q(t)$ لا تكافئ الصفر.

الهدف من هذا البحث هو ايجاد شروط كافية للحلول غير المتذبذبة للمعادلة (1.1) تضمن تقاربها الى الصفر أو تكون متباعدة ، حيث تم مناقشة السلوك المحاذي لحلول المعادلة (1.1) ومعرفة الدوال المؤثرة على استقراره أو تباعده . يتضمن البحث أربع نظريات ونتيجتين ،كما وتضمن أيضا أمثلة توضيحية للنتائج التي تم الحصول عليها للتأكيد على أن مجموعة الحلول التي تحقق الشروط المستخرجة غير خالية.