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Best Proximity Point Theorem for $\tilde{\alpha}-\tilde{\psi}$ −Contractive Type Mapping in Fuzzy Normed Space

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Abstract:

The best proximity point is a generalization of a fixed point that is beneficial when the contraction map is not a self-map. On other hand, best approximation theorems offer an approximate solution to the fixed point equation $\mathbb{T}p = p$. It is used to solve the problem in order to come up with a good approximation. This paper's main purpose is to introduce new types of proximal contraction for nonself mappings in fuzzy normed space and then proved the best proximity point theorem for these mappings. At first, the definition of fuzzy normed space is given. Then the notions of the best proximity point and $\tilde{\alpha}$- proximal admissible in the context of fuzzy normed space are presented. The notion of $\tilde{\alpha}-\tilde{\psi}$- proximal contractive mapping is introduced. After that, the best proximity point theorem for such type of mapping in a fuzzy normed space is state and prove. In addition, the idea of $\tilde{\alpha}-\tilde{\phi}$-proximal contractive mapping is presented in a fuzzy normed space and under specific conditions, the best proximity point theorem for such type of mappings is proved. Furthermore, some examples are offered to show the results' usefulness.

Keywords: Best proximity point, Fuzzy normed space, $\tilde{\alpha}$- Proximal admissible mapping, $\tilde{\alpha}-\tilde{\psi}$-Proximal contractive mapping, $\tilde{\alpha}-\tilde{\phi}$-Proximal contractive.

Introduction:

Zadeh [1] proposed and investigated the idea of a fuzzy set in his fundamental paper. The research of fuzzy sets led to the fuzzification of a variety of mathematical notions, and it may be used in a variety of fields. Kramosil and Michalek [2] were the first to establish the notion of fuzzy metric spaces. George and Veeramani [3] modified the notion of fuzzy metric spaces. A wide number of works have been published in fuzzy metric spaces; see [4-7]. Katsaras A, [8] was the first to establish the fuzzy norm on a linear space. A considerable of papers for the fuzzy normed spaces were published, for example, see [9-12]. The best approximation theorems provide an approximate solution to the fixed-point equation $\mathbb{T}p = p$ when the non-self mapping $\mathbb{T}$ has no fixed point. Particularly, a known best approximation theorem, attributed to Fan K[13], states that if $\widetilde{W}$ represents a Hausdorff locally convex topological vector space and $\widetilde{U}$ is a subset of $\widetilde{W}$ where $\widetilde{U}$ is a nonempty compact convex set and mapping $\mathbb{T}: \widetilde{U} \to \widetilde{W}$ is continuous, then there exists an element a satisfying the condition $m(a, \mathbb{T}a) = inf\, m(b, \mathbb{T}a): b \in \widetilde{U}$, where m is a metric on $\widetilde{W}$.

The Best proximity point evolves as a generalization of the concept of best approximation. Precisely, although the best approximation theorem ($\mathcal{BPP}$-theorem) guarantees the existence of an approximate solution, a best proximity point theorem is contemplated for solving the problem to find an approximate solution that is optimum. Let $\widetilde{U}$ and $\widetilde{V}$ be nonempty closed subsets of $\widetilde{W}$, when a nonself-mapping $\mathbb{T}: \widetilde{U} \to \widetilde{V}$ does not possess a fixed point, it is quite natural to find an element $a^\star$ such that $m(a^\star, \mathbb{T}a^\star)$ is minimal. $\mathcal{BPP}$-theorems ensure that an element $a^\star$ exists where $m(a^\star, \mathbb{T}a^\star) = m(\widetilde{U}, \widetilde{V}) = inf\, m(a, b): a \in \widetilde{U}, b \in \widetilde{V}$. This element is called the best proximity point of $\mathbb{T}$. Furthermore, when the mapping in question is self-mapping, $\mathcal{BPP}$-theorem yields a fixed point result. References[14-17] and the references therein provide some results in this approach.

In this work, the notion of $\tilde{\alpha}$- proximal admissible, $\tilde{\alpha}$–$\tilde{\psi}$- proximal contractive and $\tilde{\alpha}$–$\tilde{\phi}$-proximal contractive is introduced for nonself mappings $\mathbb{T}: \tilde{U} \to \tilde{V}$ and $\mathcal{BPP}$-theorem for these types of mappings is proved.

Preliminaries:

This section defines the terminology and outcomes which is going to be utilized throughout the paper.

Definition 1: [18] Let L be a vector space over a field R. A triplet $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ is termed as fuzzy normed space (briefly, $\mathcal{F}_{\mathcal{N}}$ space) where $\otimes$ is a t-norm and $\mathcal{F}_{\mathcal{N}}$ is a fuzzy set on $L \times R$ that meets the conditions below for all $p, q \in L$:

$(\mathcal{F}_{\mathcal{N}\,1})$ $\mathcal{F}_{\mathcal{N}}(p, 0) = 0$,

$(\mathcal{F}_{\mathcal{N}\,2})$ $\mathcal{F}_{\mathcal{N}}(p, \tau) = 1, \forall \tau > 0$ if and only if $P = 0$,

$(\mathcal{F}_{\mathcal{N}\,3})$ $\mathcal{F}_{\mathcal{N}}(\gamma p, \tau) = \mathcal{F}_{\mathcal{N}}(p, \tau/|\gamma|), \forall\ (0 \neq)\gamma \in R, \tau \geq 0$

$(\mathcal{F}_{\mathcal{N}\,4})$ $\mathcal{F}_{\mathcal{N}}(p, \tau) \otimes \mathcal{F}_{\mathcal{N}}(q, s) \leq \mathcal{F}_{\mathcal{N}}(p + q, \tau + s), \forall \tau, s \geq 0$

$(\mathcal{F}_{\mathcal{N}\,5})$ $\mathcal{F}_{\mathcal{N}}(p, .)$ is left continuous for each $p \in L$, and $\lim_{\tau \to \infty} \mathcal{F}_{\mathcal{N}}(p, \tau) = 1$.

Definition 2: [19] Let $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ be a $\mathcal{F}_{\mathcal{N}}$ space. Then

(1) a sequence $\{p_n\}$ is termed as a convergent sequence if $\lim_{\tau \to \infty} \mathcal{F}_{\mathcal{N}}(p_n - p, \tau) = 1$ for each $\tau > 0$ and $p \in L$.

(2) a sequence $\{p_n\}$ is termed as a Cauchy if $\lim_{n \to \infty} \mathcal{F}_{\mathcal{N}}\left(p_{n+j} - p_n, \tau\right) = 1$; for each $\tau > 0$ and $j = 1,2, \ldots$

Definition 3: [19] Let $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ be a $\mathcal{F}_{\mathcal{N}}$ space. Then $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ is termed as complete if every Cauchy sequence in L is convergent in L.

In a fuzzy metric space $(L, \mathcal{F}_{\mathcal{M}}, \otimes)$, Vetro and Salimi [20] presented the notion of fuzzy distance. Consider $\tilde{U}$ *and* $\tilde{V}$ be nonempty subsets of $(L, \mathcal{F}_{\mathcal{M}}, \otimes)$ and $\tilde{U}_{\circ}(\tau)$, $\tilde{V}_{\circ}(\tau)$ denoted by the following sets:

$\tilde{U}_{\circ}(\tau) = \{p \in \tilde{U} : \mathcal{F}_{\mathcal{M}}(p, q, \tau) = \mathcal{F}_{\mathcal{M}}(\tilde{U}, \tilde{V}, \tau)$ for some $q \in \tilde{V}\}$

$\tilde{V}_{\circ}(\tau) = \{q \in \tilde{V} : \mathcal{F}_{\mathcal{M}}(p, q, \tau) = \mathcal{F}_{\mathcal{M}}(\tilde{U}, \tilde{V}, \tau)$ for some $p \in \tilde{U}\}$

where $\mathcal{F}_{\mathcal{M}}\left(\tilde{U}, \tilde{V}, \tau\right) = \sup\{\mathcal{F}_{\mathcal{M}}(p, q, \tau): p \in \tilde{U}, q \in \tilde{V}\}$,

In this paper, the above notion is introduced in a $\mathcal{F}_{\mathcal{N}}$ space as follows:

Consider $\tilde{U}$ and $\tilde{V}$ be nonempty subsets of $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ and $\tilde{U}_{\circ}(\tau)$, $\tilde{V}_{\circ}(\tau)$ denoted by the following sets:

$\tilde{U}_{\circ}(\tau) = \{p \in U : \mathcal{F}_{\mathcal{N}}(p - q, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)$ for some $q \in \tilde{V}\}$;

$\tilde{V}_{\circ}(\tau) = \{q \in \tilde{V} : \mathcal{F}_{\mathcal{N}}(p - q, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)$ for some $p \in \tilde{\mathbb{U}}\}$;

where $N_d(\tilde{\mathbb{U}}, \tilde{V}, \tau) = sup\{\mathcal{F}_{\mathcal{N}}(p - q, \tau): p \in \tilde{U}, q \in \tilde{V}\}$.

Main Results

In this section, $\tilde{\alpha}$-proximal admissible, $\tilde{\alpha}$–$\tilde{\psi}$-proximal contractive and $\tilde{\alpha}$–$\tilde{\phi}$-proximal contractive mappings are defined, then our main results are proved.

In a fuzzy metric space, Saha et al.[21] proposed the notion of $\mathcal{BPP}$. In the following, the notion of the $\mathcal{BPP}$ in the context of $\mathcal{F}_{\mathcal{N}}$ space is introduced.

Definition 4: Let $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ be a fuzzy Banach space and $\tilde{U}$, $\tilde{V}$ are nonempty closed subsets of L An element $p^{\star} \in \tilde{U}$ is called the best proximity point $(\mathcal{BPP})$ of a mapping $\mathbb{T}: \tilde{U} \to \tilde{V}$ if $\mathcal{F}_{\mathcal{N}}(p^{\star} - \mathbb{T}p^{\star}, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)$ for all $\tau > 0$.

Next, the definition of $\tilde{\alpha}$- proximal admissible and $\tilde{\alpha}$–$\tilde{\psi}$- proximal contractive mapping is presented.

Let $\tilde{\Psi}$ represent the collection of all functions $\tilde{\psi}: [0, 1] \to [0, 1]$, having the following properties:

1) $\tilde{\psi}$ decreasing and for all $\mu \in [0, 1]$, $\tilde{\psi}(\mu) > \mu$
2) $\tilde{\psi}$ continuous
3) $\tilde{\psi}(\mu) = 1$ if and only if $\mu = 1$.

Definition 5.: Let $\tilde{U}$ and $\tilde{V}$ be two nonempty subsets of a $\mathcal{F}_{\mathcal{N}}$ space $(L, \mathcal{F}_{\mathcal{N}}\ \otimes)$ and $\mathbb{T}: \tilde{U} \to \tilde{V}$ is a mapping. Then $\mathbb{T}$ is termed as an $\tilde{\alpha}$- proximal admissible mapping where $\tilde{\alpha} : \tilde{U} \times \tilde{U} \times [0, \infty) \to [0, \infty)$ if for each $p, q, u, v \in \tilde{U}$, and $\tau > 0$

$$\left.\begin{array}{l}\tilde{\alpha}(p, q, \tau) \leq 1 \\ \mathcal{F}_{\mathcal{N}}(u - \mathbb{T}p, \tau) = N_d(\tilde{U}, \tilde{V}, \tau) \\ \mathcal{F}_{\mathcal{N}}(v - \mathbb{T}q, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)\end{array}\right\} \Rightarrow \tilde{\alpha}(u, v, \tau) \leq 1 \qquad 1$$

Definition 6.: Let $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ be a $\mathcal{F}_{\mathcal{N}}$ space and $\mathbb{T}: \tilde{U} \to \tilde{V}$ is a mapping where $\tilde{U}$, $\tilde{V}$ are two nonempty subsets of L. Then $\mathbb{T}$ is termed as an $\tilde{\alpha}$–$\tilde{\psi}$- proximal contractive mapping where $\tilde{\alpha} : \tilde{U} \times \tilde{U} \times [0, \infty) \to [0, \infty)$ if there exists a function $\tilde{\psi} \in \tilde{\Psi}$ such that

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$$\left.\begin{array}{r}\tilde{\alpha}(\mathrm{p},\mathrm{q},\tau) \leq 1 \\ \mathcal{F}_{\mathcal{N}}(u - \mathbb{T}x,\tau) = N_d(\widetilde{U},\widetilde{V},\tau) \\ \mathcal{F}_{\mathcal{N}}(v - \mathbb{T}\mathrm{q},\tau) = N_d(\widetilde{U},\widetilde{V},\tau)\end{array}\right\} \Rightarrow$$
$$\tilde{\alpha}(\mathrm{p},\mathrm{q},\tau)\mathcal{F}_{\mathcal{N}}(u - v,\tau) \geq \tilde{\psi}(\mathcal{F}_{\mathcal{N}}(\mathrm{p} - \mathrm{q},\tau)) \quad 2$$

for each $\mathrm{p},\mathrm{q},u,v \in \widetilde{U}$, and $\tau > 0$.

Theorem 1: Assume that $(L,\mathcal{F}_{\mathcal{N}}, \otimes)$ be a fuzzy Banach space and let $\widetilde{U}$ and $\widetilde{V}$ nonempty closed subsets of L where $\widetilde{U}_\circ(\tau)$ is nonempty for each $\tau > 0$. Consider $\mathbb{T} : \widetilde{U} \to \widetilde{V}$ is an $\tilde{\alpha}$–$\tilde{\psi}$- proximal contractive mapping meeting the following conditions:

(a) $\mathbb{T}$ is $\tilde{\alpha}$- proximal admissible mapping and $\mathbb{T}(\widetilde{U}_\circ(\tau)) \subseteq \widetilde{V}_\circ(\tau)$ for each $\tau > 0$.
(b) In $\widetilde{U}_\circ(\tau)$ there are elements $\mathrm{p}_\circ$ and p_1 such that $\mathcal{F}_{\mathcal{N}}(\mathrm{p}_1 - \mathbb{T}\mathrm{p}_\circ,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$; $\tilde{\alpha}(\mathrm{p}_\circ,\mathrm{p}_1,\tau) \leq 1$ for each $\tau > 0$.
(c) If $\{\mathrm{q}_n\}$ is a sequence in $\widetilde{V}_\circ(\tau)$ and $\mathrm{p} \in \widetilde{U}$ such that $\mathcal{F}_{\mathcal{N}}(\mathrm{p} - \mathrm{q}_n,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$ as $n \to \infty$, then $\mathrm{p} \in \widetilde{U}_\circ(\tau)$ for each $\tau > 0$.
(d) If $\{\mathrm{p}_n\}$ is a sequence in L such that $\tilde{\alpha}(\mathrm{p}_n,\mathrm{p}_{n+1},\tau) \leq 1$, $\forall n \geq 1$ and $\mathrm{p}_n \to \mathrm{p}$ as $n \to \infty$, then $\tilde{\alpha}(\mathrm{p}_n,\mathrm{p},\tau) \leq 1 \ \forall n \geq 1$ and $\tau > 0$.
(e) Moreover, if $\mathcal{F}_{\mathcal{N}}(\mathrm{p} - \mathbb{T}\mathrm{p},\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$ and $\mathcal{F}_{\mathcal{N}}$ (q - $\mathbb{T}$ q, τ) = $N_d(\widetilde{U},\widetilde{V},\tau)$ implies that $\tilde{\alpha}(\mathrm{p},\mathrm{q},\tau) \leq 1$ for each $\tau > 0$, then $\mathbb{T}$ possess a unique $\mathcal{BPP}$.

proof: According to condition (b), there are elements, say $\mathrm{p}_\circ$, p_1 in $\widetilde{U}_\circ(\tau)$ such that
$\mathcal{F}_{\mathcal{N}}(\mathrm{p}_1 - \mathbb{T}\mathrm{p}_\circ,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$; $\tilde{\alpha}(\mathrm{p}_\circ,\mathrm{p}_1,\tau) \leq 1$ for each $\tau > 0$.

Since $\mathbb{T}(\widetilde{U}_\circ(\tau)) \subseteq \widetilde{V}_\circ(\tau)$, there exists $\mathrm{p}_2 \in \widetilde{U}_\circ(\tau)$ such that

$$\mathcal{F}_{\mathcal{N}}(\mathrm{p}_2 - \mathbb{T}\mathrm{p}_1,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$$

Because $\mathbb{T}$ is an $\tilde{\alpha}$- proximal admissible mapping, then $\tilde{\alpha}(\mathrm{p}_1,\mathrm{p}_2,\tau) \leq 1$.

Again, since $\mathbb{T}(\widetilde{U}_\circ(\tau)) \subseteq \widetilde{V}_\circ(\tau)$, there is $\mathrm{p}_3 \in \widetilde{U}_\circ(\tau)$ such that

$$\mathcal{F}_{\mathcal{N}}(\mathrm{p}_3 - \mathbb{T}\mathrm{p}_2,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$$

Thus

$$\mathcal{F}_{\mathcal{N}}(\mathrm{p}_2 - \mathbb{T}\mathrm{p}_1,\tau) = N_d(\widetilde{U},\widetilde{V},\tau);$$
$$\mathcal{F}_{\mathcal{N}}(\mathrm{p}_3 - \mathbb{T}\mathrm{p}_2,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$$
$$\tilde{\alpha}(\mathrm{p}_1,\mathrm{p}_2,\tau) \leq 1$$

and because $\mathbb{T}$ is $\tilde{\alpha}$–proximal admissible mapping, then $\tilde{\alpha}(\mathrm{p}_2,\mathrm{p}_3,\tau) \leq 1$.

If we keep going this way, obtain:

$$\mathcal{F}_{\mathcal{N}}(\mathrm{p}_{\mathrm{n}+1} - \mathbb{T}\mathrm{p}_{\mathrm{n}},\tau) = N_d(\widetilde{U},\widetilde{V},\tau);$$
$$\tilde{\alpha}(\mathrm{p}_n,\mathrm{p}_m,\tau) \leq 1 \quad 3$$

Now using Eq.3 and applying the inequality 2 with $u = \mathrm{q} = \mathrm{p}_n$, $v = \mathrm{p}_{n+1}$ and $\mathrm{p} = \mathrm{p}_{n-1}$ obtain:

$$\begin{aligned}\mathcal{F}_{\mathcal{N}}(\mathrm{p}_{\mathrm{n}} - \mathrm{p}_{\mathrm{n}+1},\tau) &\geq \tilde{\alpha}(\mathrm{p}_{n-1},\mathrm{p}_n,\tau)\mathcal{F}_{\mathcal{N}}(\mathbb{T}\mathrm{p}_{n-1} - \mathbb{T}\mathrm{p}_n,\tau) \\ &\geq \tilde{\psi}(\mathcal{F}_{\mathcal{N}}(\mathrm{p}_{n-1} - \mathrm{p}_n,\tau))\end{aligned}$$

Hence
$$\mathcal{F}_{\mathcal{N}}(p_n - p_{n+1},\tau) \geq \tilde{\psi}\big(\mathcal{F}_{\mathcal{N}}(p_{n-1} - p_n,\tau)\big) \quad 4$$

and hence $\{\mathcal{F}_{\mathcal{N}}(\mathrm{p}_{\mathrm{n}} - \mathrm{p}_{\mathrm{n}+1},\tau)\}$ in (0,1] is an increasing sequence, consequently, there is $\gamma(\tau) \in (0,1]$ such that $\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}(p_n - p_{n+1},\tau) = \gamma(\tau)$ $\forall \tau > 0$.

Now, it will be established that $\gamma(\tau) = 1$ for each $\tau > 0$. Assume that there is $\tau_\circ > 0$ such that $0 < \gamma(\tau_\circ) < 1$. Passing to limit as $n \to \infty$ in inequality 4, obtain

$$\gamma(\tau_\circ) \geq \tilde{\psi}(\ell(\tau_\circ))$$

If $\tilde{\psi}(\gamma(\tau_\circ)) = 1$ then there's a contradiction. Hence $\gamma(\tau) = 1$ and conclude that

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}(\mathrm{p}_{\mathrm{n}} - \mathrm{p}_{\mathrm{n}+1},\tau) = 1 \quad \forall \tau > 0 \quad 5$$

Following that, to show that $\{\mathrm{p}_{\mathrm{n}}\}$ is a Cauchy sequence. Consider $\{\mathrm{p}_{\mathrm{n}}\}$ is not Cauchy. Then there is $\mathfrak{z} \in (0,1)$ such that for all $\kappa \geq 1$, there are $\mathfrak{m}(\kappa),\mathfrak{n}(\kappa) \in N$ with $\mathfrak{m}(\kappa) > \mathfrak{n}(\kappa) \geq \kappa$ and

$$\mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathfrak{n}(\kappa)},\tau_\circ\right) \leq 1 - \mathfrak{z} \quad ,\tau_\circ > 0$$

Assume that $\mathfrak{m}(\kappa)$ is the smallest integer greater than $\mathfrak{n}(\kappa)$, meeting the condition above,

$$\mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)-1} - \mathrm{p}_{\mathfrak{n}(\kappa)},\tau_\circ\right) > 1 - \mathfrak{z}$$

and for all κ,

$$\begin{aligned}1 - \mathfrak{z} &\geq \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathfrak{n}(\kappa)},\tau_\circ\right) \\ &\geq \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathfrak{m}(\kappa)-1},\tau_\circ\right) \\ &\qquad \otimes \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)-1} - \mathrm{p}_{\mathfrak{n}(\kappa)},\tau_\circ\right)\end{aligned}$$

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$$> \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathfrak{m}(\kappa)-1}, \tau_\circ\right) \otimes 1 - \mathfrak{z}$$

In the previous inequality, if use limit as $\kappa \to \infty$ and using Eq.5, obtained:

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathfrak{n}(\kappa)}, \tau_\circ\right) = 1 - \mathfrak{z} \quad 6$$

Now from

$$\begin{aligned}\mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)+1} - p_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right) &\geq \mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)+1} - p_{\mathfrak{m}(\kappa)}, \tau_\circ\right) \\ &\otimes \mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)} - p_{\mathfrak{n}(\kappa)}, \tau_\circ\right) \\ &\otimes \mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{n}(\kappa)} - p_{n(\kappa)+1}, \tau_\circ\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)} - p_{\mathfrak{n}(\kappa)}, \tau_\circ\right) &\geq \mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)} - p_{\mathfrak{m}(\kappa)+1}, \tau_\circ\right) \\ &\otimes \mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)+1} - p_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right) \\ &\otimes \mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{n}(\kappa)+1} - p_{\mathfrak{n}(\kappa)}, \tau_\circ\right)\end{aligned}$$

it follows that

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)+1} - \mathrm{p}_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right) = 1 - \mathfrak{z} \quad 7$$

From Eq.3,

$$\begin{cases}\tilde{\alpha}(\mathrm{p}_{\mathfrak{n}(\kappa)}, \mathrm{p}_{\mathfrak{m}(k)}, \tau_\circ) \leq 1 \\ \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)+1} - \mathbb{T}\mathrm{p}_{\mathfrak{m}(\kappa)}, \tau_\circ\right) = \mathrm{N_d}(\tilde{\mathrm{U}}, \tilde{\mathrm{V}}, \tau_\circ) \\ \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{n}(\kappa)+1} - \mathbb{T}\mathrm{p}_{\mathfrak{n}(\kappa)}, \tau_\circ\right) = \mathrm{N_d}(\tilde{\mathrm{U}}, \tilde{\mathrm{V}}, \tau_\circ)\end{cases} \quad 8$$

Hence, by inequality 2 and Eq. 8:

$$\tilde{\alpha}(\mathrm{p}_{\mathfrak{n}(\kappa)}, \mathrm{p}_{\mathfrak{m}(\kappa)}, \tau_\circ)\mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)+1} - \mathrm{p}_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right) \geq \tilde{\psi}(\mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathfrak{n}(\kappa)}, \tau_\circ\right))$$

In the previous inequality, if use limit as $\kappa \to \infty$, obtain:

$$1 - \mathfrak{z} \geq \tilde{\psi}(1 - \mathfrak{z})$$

and this is a contradiction. Also, if $\tilde{\psi}(1 - \mathfrak{z}) = 1$, then, by property (3) of $\tilde{\psi}$, $\mathfrak{z} = 0$ but this contradiction, therefore $\{\mathrm{p}_n\}$ is a Cauchy. Because $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ is complete then $\{\mathrm{p}_n\}$ converges to some $\mathrm{p}^* \in L$,

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}(p_n - p^\star, \tau) = 1 \quad \text{for each } \tau > 0.$$

Furthermore,

$$\begin{aligned}N_d(\tilde{U}, \tilde{V}, \tau) &= \mathcal{F}_{\mathcal{N}}(\mathrm{p_{n+1}} - \mathbb{T}\mathrm{p_n}, \tau) \quad \text{(by Eq.3)} \\ &\geq \mathcal{F}_{\mathcal{N}}(\mathrm{p_{n+1}} - \mathrm{p}^*, \tau) \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \mathbb{T}\mathrm{p_n}, \tau) \quad \text{(applying condition } (\mathcal{F}_{\mathcal{N}\,4})\text{)} \\ &\geq \mathcal{F}_{\mathcal{N}}(\mathrm{p_{n+1}} - \mathrm{p}^*, \tau) \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p}^* - \mathrm{p_{n+1}}, \tau) \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p_{n+1}} - \mathbb{T}\mathrm{p_n}, \tau) \quad \text{(applying } (\mathcal{F}_{\mathcal{N}\,4})\text{)} \\ &= \mathcal{F}_{\mathcal{N}}(\mathrm{p_{n+1}} - \mathrm{p}^*, \tau) \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p}^* - \mathrm{p_{n+1}}, \tau) \otimes N_d(\tilde{U}, \tilde{V}, \tau)\end{aligned}$$

which implies

$$\begin{aligned}N_d(\tilde{U}, \tilde{V}, \tau) &\geq \mathcal{F}_{\mathcal{N}}(\mathrm{p_{n+1}} - \mathrm{p}^*, \tau) \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p}^* - \mathbb{T}\mathrm{p_n}, \tau) \\ &\geq \mathcal{F}_{\mathcal{N}}(\mathrm{p_{n+1}} - \mathrm{p}^*, \tau) \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p}^* - \mathrm{p_{n+1}}, \tau) \otimes N_d(\tilde{U}, \tilde{V}, \tau)\end{aligned}$$

In the previous inequality, if use limit as $\mathfrak{n} \to \infty$, obtained:

$$\begin{aligned}N_d(\tilde{U}, \tilde{V}, \tau) &\geq 1 \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p}^* - \mathbb{T}\mathrm{p_n}, \tau) \\ &\geq 1 \otimes 1 \otimes N_d(\tilde{U}, \tilde{V}, \tau)\end{aligned}$$

that is,

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}(\mathrm{p}^* - \mathbb{T}\mathrm{p_n}, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)$$

and by condition (c), $\mathrm{p}^\star \in \tilde{U}_\circ(\tau)$. Now since $\mathbb{T}(\tilde{U}_\circ(\tau)) \subseteq \tilde{V}_\circ(\tau)$, there is $z \in \tilde{U}_\circ(\tau)$ with $\mathcal{F}_{\mathcal{N}}(\mathrm{z} - \mathbb{T}\mathrm{p}^\star, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)$. Consequently, it follows from condition (d) and inequality 2 with $\mathcal{u} = \mathrm{p}_{n+1}$, $\mathcal{v} = z$, $\mathrm{p} = \mathrm{p}_n$ and $\mathrm{q} = \mathrm{p}^\star$ that

$$\tilde{\alpha}(\mathrm{p}_n, \mathrm{p}^\star, \tau)\mathcal{F}_{\mathcal{N}}(\mathrm{p}_{n+1} - \mathrm{z}, \tau) \geq \tilde{\psi}(\mathcal{F}_{\mathcal{N}}(\mathrm{p}_n - \mathrm{p}^\star, \tau)).$$

$$\Rightarrow \quad \mathcal{F}_{\mathcal{N}}(\mathrm{p}_{n+1} - \mathrm{z}, \tau) \geq \tilde{\psi}(\mathcal{F}_{\mathcal{N}}(\mathrm{p}_n - \mathrm{p}^\star, \tau))$$

In the previous inequality, if use limit as $\mathfrak{n} \to \infty$, obtain:

$$\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \mathrm{z}, \tau) = 1 \quad \text{for each } \tau > 0$$

Therefore $\mathrm{p}^\star = \mathrm{z}$ and $\mathcal{F}_{\mathcal{N}}(p^\star - \mathbb{T}p^\star, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)$.

Now to prove that $\mathrm{p}^\star$ is a unique $\mathcal{BPP}$ of $\mathbb{T}$. Consider ω other $\mathcal{BPP}$ of $\mathbb{T}$, $\omega \neq \mathrm{p}^\star$ that is $\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \mathbb{T}\mathrm{p}^\star, \tau) = N_d(\tilde{U}, \tilde{V}, \tau)$ and $\mathcal{F}_{\mathcal{N}}(\omega - \mathbb{T}\omega, \tau) = N_d\left(\tilde{U}, \tilde{V}, \tau\right)$. Now, if condition (e) of the theorem holds, then from inequality 2,

$$\tilde{\alpha}(\mathrm{p}^\star, \omega, \tau)\mathcal{F}_{\mathcal{N}}(p^\star - \omega, \tau) \geq \tilde{\psi}(\mathcal{F}_{\mathcal{N}}(p^\star - \omega, \tau))$$

$$\Rightarrow \quad \mathcal{F}_{\mathcal{N}}(p^\star - \omega, \tau) \geq \tilde{\psi}\left(\mathcal{F}_{\mathcal{N}}(p^\star - \omega, \tau)\right)$$

which is a contradiction with property 1 of $\tilde{\psi}$ and hence $\mathcal{F}_{\mathcal{N}}\,(p^{\star}-\omega,\tau)=1$ for each $\tau>0$, that is, $p^{\star}=\omega$.

Example 1: Let $L=\mathbb{R}$ with the fuzzy norm, $\mathcal{F}_{\mathcal{N}}: L\times\mathbb{R}\to[0,1]$ define by

$\mathcal{F}_{\mathcal{N}}\,(p,\tau)=\frac{\tau}{\tau+\|p\|}$, $\forall p\in L$ and $\tau>0$, where $\|p\|=|p|$.

Let $\tilde{\psi}:[0,1]\to[0,1]$ with the property $\tilde{\psi}(1)=1$. Suppose that $\tilde{U}$ *and* $\tilde{V}$ are nonempty subsets of L specified by:

$\tilde{U}=\{0,1,\frac{1}{2},\frac{1}{4},\frac{2}{5}\}$ and $\tilde{V}=\{0,1,\frac{1}{3},\frac{1}{5},\frac{7}{9}\}$.

Note that $N_d(\tilde{U},\tilde{V},\tau)=1$, so $\tilde{U}_{\circ}(\tau)=\{0,1\}$ and $\tilde{V}_{\circ}(\tau)=\{0,1\}$

We defined $\mathbb{T}:\tilde{U}\to\tilde{V}$ as follow:

$\mathbb{T}p=1$ for all $p\in\tilde{U}$

while $\tilde{\alpha}:\tilde{U}\times\tilde{U}\times[0,\infty)\to[0,\infty)$ specified as :

$\tilde{\alpha}(p,q,\tau)=1,\ \forall p,q\in\tilde{U}$.

Clearly, $\mathbb{T}(\tilde{U}_{\circ}(\tau))\subseteq\tilde{V}_{\circ}(\tau)$.

Assume that $\mathcal{F}_{\mathcal{N}}\,(u-\mathbb{T}p,\tau)=N_d(\tilde{U},\tilde{V},\tau)$and $\mathcal{F}_{\mathcal{N}}\,(v-\mathbb{T}q,\tau)=N_d(\tilde{U},\tilde{V},\tau)$ for some $u,v,p,q\in\tilde{U}$.

Assume that

$$\begin{cases}\tilde{\alpha}(p,q,\tau)\le 1\\ \mathcal{F}_{\mathcal{N}}\,(u-\mathbb{T}p,\tau)=N_d(\tilde{U},\tilde{V},\tau)\\ \mathcal{F}_{\mathcal{N}}\,(v-\mathbb{T}q,\tau)=N_d(\tilde{U},\tilde{V},\tau)\end{cases}$$

Then

$$\begin{cases}p,q\in\tilde{U}\\ \mathcal{F}_{\mathcal{N}}\,(u-\mathbb{T}p,\tau)=N_d(\tilde{U},\tilde{V},\tau)\\ \mathcal{F}_{\mathcal{N}}\,(v-\mathbb{T}q,\tau)=N_d(\tilde{U},\tilde{V},\tau)\end{cases}$$

Hence $u=v=1$ that is $\tilde{\alpha}(u,v,\tau)\le 1$, which means $\mathbb{T}$ is an $\tilde{\alpha}$ −proximal admissible mapping.
$\mathbb{T}$ is an $\tilde{\alpha}$–$\tilde{\psi}$- proximal contractive mapping with - $\tilde{\psi}(\mu)=\sqrt{\mu}$, $\forall\mu\in[0,1]$. In effect, for each p, q $\in\tilde{U}$,

$$\tilde{\alpha}(p,q,\tau)\mathcal{F}_{\mathcal{N}}\,(u-v,\tau)\ge\tilde{\psi}\big(\mathcal{F}_{\mathcal{N}}\,(p-q,\tau)\big)$$

Thus each of Theorem 1's hypotheses is fulfilled. As a result, $\mathbb{T}$ possesses a unique $\mathcal{BPP}$. $p^{\star}=1$ represents a unique $\mathcal{BPP}$ in this example

In the following, the definition of $\tilde{\alpha}$–$\tilde{\phi}$- proximal contractive for mappings $\mathbb{T}:\tilde{U}\to\tilde{V}$ is presented and the $\mathcal{BPP}$-theorem is introduced for this type of mapping.

Definition 7: Let $(L,\mathcal{F}_{\mathcal{N}},\otimes)$ be a $\mathcal{F}_{\mathcal{N}}$ space and let $\tilde{U}$, $\tilde{V}$ be two nonempty subsets of L**.** Assume that $\mathbb{T}:\tilde{U}\to\tilde{V}$ be a given mapping. Then $\mathbb{T}$ is termed as $\tilde{\alpha}$–$\tilde{\phi}$- proximal contractive mapping where $\tilde{\alpha}:\tilde{U}\times\tilde{U}\times[0,\infty)\to[0,\infty)$ if for each $p,q,u,v\in\tilde{U}$, and $\tau>0$**,**

$$\left.\begin{matrix}\tilde{\alpha}(p,q,\tau)\le 1\\ \mathcal{F}_{\mathcal{N}}\,(u-\mathbb{T}p,\tau)=N_d(\tilde{U},\tilde{V},\tau)\\ \mathcal{F}_{\mathcal{N}}\,(v-\mathbb{T}q,\tau)=N_d(\tilde{U},\tilde{V},\tau)\end{matrix}\right\}\Rightarrow\mathcal{F}_{\mathcal{N}}\,(u-v,\tau)\ge\mathcal{F}_{\mathcal{N}}\,(p-q,\tau)+\tilde{\phi}\big(\mathfrak{B}(p\,q,u,v,\tau)\big)\quad 9$$

where $\mathfrak{B}(p\,q,u,v,\tau)=\min\{\mathcal{F}_{\mathcal{N}}\,(p-q,\tau),\max\{\mathcal{F}_{\mathcal{N}}\,(p-u,\tau),\mathcal{F}_{\mathcal{N}}\,(q-v,\tau)\}\}$and $\tilde{\phi}:[0,1]\to[0,1]$ is continuous and for each $\mu\in(0,1)$, $\tilde{\phi}(\mu)>0$.

Theorem 2: Assume that $(L,\mathcal{F}_{\mathcal{N}},\otimes)$ be a fuzzy Banach space and $\tilde{U}$, $\tilde{V}$ be nonempty closed subsets of L where $\tilde{U}_{\circ}(\tau)$ is nonempty for each $\tau>0$. Consider $\mathbb{T}:\tilde{U}\to\tilde{V}$ be an $\tilde{\alpha}$–$\tilde{\phi}$-proximal contractive mapping meeting the following conditions:

(a) T is $\tilde{\alpha}$ - proximal admissible mapping and $\mathbb{T}(\tilde{U}_{\circ}(\tau))\subseteq\tilde{V}_{\circ}(\tau)$,$\forall\tau>0$;
(b) In $\tilde{U}_{\circ}(\tau)$ there are elements $p_{\circ}$ and p_1 with $\mathcal{F}_{\mathcal{N}}\,(p_1-\mathbb{T}p_{\circ},\tau)=N_d(\tilde{U},\tilde{V},\tau)$; $\tilde{\alpha}(p_{\circ},p_1,\tau)\le 1$ for all $\tau>0$;
(c) If $\{q_n\}$ is a sequence in $\tilde{V}_{\circ}(\tau)$ and $p\in\tilde{U}$ is such that $\mathcal{F}_{\mathcal{N}}\,(p-q_n,\tau)=N_d(\tilde{U},\tilde{V},\tau)$ as $n\to\infty$, then $p\in\tilde{U}_{\circ}(\tau)$ for all $\tau>0$.
(d) If $\{p_n\}$ is a sequence in L such that $\tilde{\alpha}\,(p_n,p_{n+1},\tau)\le 1$ for all $n\ge 1$ and $p_n\to p$ as $n\to\infty$, then $\tilde{\alpha}\,(p_n,p,\tau)\le 1$ for all $n\ge 1$ and $\tau>0$.
(e) Moreover, if $\mathcal{F}_{\mathcal{N}}\,(p-\mathbb{T}p,\tau)=N_d(\tilde{U},\tilde{V},\tau)$ and $\mathcal{F}_{\mathcal{N}}$ (q- $\mathbb{T}$q, τ) $=N_d(\tilde{U},\tilde{V},\tau)$ implies that $\tilde{\alpha}\,(p,q,\tau)\le 1$ for each $\tau>0$, then $\mathbb{T}$ possesses a unique $\mathcal{BPP}$.

proof: By using a similar approach as in proving Theorem1, a sequence $\{p_n\}$ in $\tilde{U}_{\circ}(\tau)$ may construct such that

$\mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathbb{T}\mathrm{p_n}, \tau) = N_d(\widetilde{U}, \widetilde{V}, \tau)$;
$\tilde{\alpha}(\mathrm{p}_n, \mathrm{p}_m, \tau) \le 1$ 10
for each $\mathfrak{n}, \mathfrak{m} \ge 1$ with $\mathfrak{n} < \mathfrak{m}$, $\tau > 0$.

Now using Eq.10 and applying the inequality 9 with $= \mathrm{q} = \mathrm{p}_n$, $\mathrm{p} = \mathrm{p}_{n-1}$ and $\mathscr{v} = \mathrm{p}_{n+1}$ obtain:
$\mathcal{F}_{\mathcal{N}}\,(p_n - p_{n+1}, \tau) \ge \mathcal{F}_{\mathcal{N}}\,(p_{n-1} - p_n, \tau) + \tilde{\phi}\big(\mathfrak{B}(p_{n-1}, p_n, p_n, p_{n+1}, \tau)\big)$
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On other hand,

$$\mathfrak{B}(\mathrm{p}_{n-1}, \mathrm{p}_n, \mathrm{p}_n, \mathrm{p}_{n+1}, \tau) = \min\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{n-1} - \mathrm{p}_n, \tau), \max\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{n-1} - \mathrm{p}_n, \tau), \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_n} - \mathrm{p}_{n+1}, \tau)\}\}$$

If $\mathcal{F}_{\mathcal{N}}\,(p_{n-1} - p_n, \tau) \le \mathcal{F}_{\mathcal{N}}\,(p_n - p_{n+1}, \tau)$ for some $n \in \mathrm{N}$, then obtain that

$$\min\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{\mathfrak{n}-1} - \mathrm{p}_n, \tau), \max\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{\mathfrak{n}-1} - \mathrm{p}_n, \tau), \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{\mathfrak{n}} - \mathrm{p}_{\mathfrak{n}+1}, \tau)\} = \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{\mathfrak{n}-1} - \mathrm{p}_n, \tau)$$

Also if $\mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{n+1} - \mathrm{p}_n, \tau) < \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_n - \mathrm{p}_{n+1}, \tau)$ for some $n \in N$, then

$$\min\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n-1}} - \mathrm{p_n}, \tau), \max\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n-1}} - \mathrm{p_n}, \tau), \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_n} - \mathrm{p_{n+1}}, \tau)\} = \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{\mathfrak{n}-1} - \mathrm{p}_{\mathfrak{n}}, \tau)$$

That is, for each $\mathfrak{n} \in N$ and $\tau > 0$,

$$\min\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p}_{\mathfrak{n}-1} - \mathrm{p_n}, \tau), \max\{\mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n-1}} - \mathrm{p_n}, \tau), \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_n} - \mathrm{p_{n+1}}, \tau)\} = \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n-1}} - \mathrm{p_n}, \tau)$$

Hence,

$$\mathcal{F}_{\mathcal{N}}\,(\mathrm{p_n} - \mathrm{p_{n+1}}, \tau) \ge \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n-1}} - \mathrm{p_n}, \tau) + \tilde{\phi}\big(\mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n-1}} - \mathrm{p_n}, \tau)\big) \qquad 12$$

which implies
$$\mathcal{F}_{\mathcal{N}}\,(p_n - p_{n+1}, \tau) \ge \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n-1}} - p_{\mathfrak{n}}, \tau)$$

and hence $\{\mathcal{F}_{\mathcal{N}}\,(p_{\mathfrak{n}} - p_{\mathfrak{n}+1}, \tau)\}$ in $(0,1]$ is an increasing sequence. Consequently, there is $\gamma(\tau) \in (0,1]$ such that $\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_n} - \mathrm{p_{n+1}}, \tau) = \gamma(\tau)$ for each $\tau > 0$. Now, it will be shown that $\gamma(\tau) = 1$ for each $\tau > 0$. Assume that there is $\tau_\circ > 0$ such that $0 < \gamma\,(\tau_\circ) < 1$.

In inequality 12 if take the limit as $\mathfrak{n} \to \infty$, then

$$\gamma(\tau_\circ) \ge \gamma(\tau_\circ) + \tilde{\phi}(\gamma(\tau_\circ))$$

then $\tilde{\phi}\big(\gamma(\tau_\circ)\big) = 0$, but this is a contradiction, hence $\gamma\,(\tau) = 1$ for each $\tau > 0$.

Now to prove that $\{\mathrm{p_n}\}$ is Cauchy. Consider $\{\mathrm{p_n}\}$ is not Cauchy and then continue as in Theorem 1's proof, there is $\mathfrak{z} \in (0,1)$ and $\tau_\circ > 0$ such that, $\forall \kappa \ge 1$, there is $\mathfrak{m}(\kappa)$; $\mathfrak{n}(\kappa) \in N$ with $\mathfrak{m}(\kappa) > \mathfrak{n}(\kappa) \ge \kappa$ such that

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathrm{n(k)}}, \tau_\circ\right) = 1 - \mathfrak{z}$$

and

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)+1} - \mathrm{p}_{\mathrm{n(k)}+1}, \tau_\circ\right) = 1 - \mathfrak{z}$$

Hence, by Eq. 8 and inequality 9,

$$\mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)+1} - \mathrm{p}_{\mathrm{n(k)}+1}, \tau_\circ\right) \ge \mathcal{F}_{\mathcal{N}}\left(\mathrm{p}_{\mathfrak{m}(\kappa)} - \mathrm{p}_{\mathrm{n(k)}}, \tau_\circ\right) + \tilde{\phi}\left(\mathfrak{B}\big(\mathrm{p}_{\mathfrak{m}(\kappa)}, \mathrm{p}_{\mathrm{n(k)}}, \mathrm{p}_{\mathfrak{m}(\kappa)+1}, \mathrm{p}_{\mathrm{n(k)}+1}, \tau_\circ\big)\right)$$
where

$$\mathfrak{B}\big(p_{\mathfrak{m}(\kappa)}, p_{n(k)}, p_{\mathfrak{m}(\kappa)+1}, p_{n(k)+1}, \tau_\circ\big) = \min\{\mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)} - p_{n(k)}, \tau_\circ\right), \max\{\mathcal{F}_{\mathcal{N}}\left(p_{\mathfrak{m}(\kappa)} - p_{\mathfrak{m}(\kappa)+1}, \tau_\circ\right), \mathcal{F}_{\mathcal{N}}\left(p_{n(k)} - p_{n(k)+1}, \tau_\circ\right)\}\}$$

by using continuity of $\tilde{\phi}$ and taking a limit as $\kappa \to \infty$ in the inequality previously, the following obtain :

$$1 - \mathfrak{z} \ge 1 - \mathfrak{z} + \tilde{\phi}\,(1 - \mathfrak{z})$$

and as a result $\tilde{\phi}(1 - \mathfrak{z}) = 0$, but this is a contradiction, hence $\{\mathrm{p}_n\}$ is a Cauchy sequence. Since $(L, \mathcal{F}_{\mathcal{N}}, \otimes)$ is complete, therefore $\{\mathrm{p}_n\}$ converges to some $\mathrm{p}^\star \in L$,

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}\,(p_n - p^\star, \tau) = 1 \qquad \text{for each } \tau > 0.$$

In addition,

$$\begin{aligned}
N_d(\widetilde{U}, \widetilde{V}, \tau) &= \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathbb{T}\mathrm{p_n}, \tau) \\
&\ge \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathrm{p}^\star, \tau) \otimes \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}^\star - \mathbb{T}\mathrm{p_n}, \tau) \\
&\ge \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathrm{p}^\star, \tau) \otimes \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}^\star - \mathrm{p_{n+1}}, \tau) \otimes \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathbb{T}\mathrm{p_n}, \tau) \\
&= \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathrm{p}^\star, \tau) \otimes \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}^\star - \mathrm{p_{n+1}}, \tau) \otimes N_d(\widetilde{U}, \widetilde{V}, \tau)
\end{aligned}$$

which implies

$$\begin{aligned}
N_d(\widetilde{U}, \widetilde{V}, \tau) &\ge \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathrm{p}^\star, \tau) \otimes \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}^\star - \mathbb{T}\mathrm{p_n}, \tau) \\
&\ge \mathcal{F}_{\mathcal{N}}\,(\mathrm{p_{n+1}} - \mathrm{p}^\star, \tau) \otimes \mathcal{F}_{\mathcal{N}}\,(\mathrm{p}^\star - \mathrm{p_{n+1}}, \tau) \otimes N_d(\widetilde{U}, \widetilde{V}, \tau)
\end{aligned}$$

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In the previous inequality, if use limit as $n \to \infty$, then:

$$N_d(\widetilde{U},\widetilde{V},\tau) \geq 1 \otimes \mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \mathbb{T}\mathrm{p}_\mathrm{n},\tau) \geq 1 \otimes 1 \otimes N_d(\widetilde{U},\widetilde{V},\tau)$$

that is,

$$\lim_{n\to\infty} \mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \mathbb{T}\mathrm{p}_\mathrm{n},\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$$

and by condition (c), $\mathrm{p}^\star \in \widetilde{U}_\circ(\tau)$. Because $\mathbb{T}(\widetilde{U}_\circ(\tau)) \subseteq \widetilde{V}_\circ(\tau)$, there is $z \in \widetilde{U}_\circ(\tau)$ with $\mathcal{F}_{\mathcal{N}}(\mathrm{z} - \mathbb{T}\mathrm{p}^\star,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$. Consequently, it follows from condition (d) and inequality 9 with $\mathcal{u} = \mathrm{p}_{n+1}$, $\mathcal{v} = z$, $\mathrm{p} = \mathrm{p}_n$ *and* $\mathrm{q} = \mathrm{p}^\star$ that

$$\mathcal{F}_{\mathcal{N}}(\mathrm{p}_{n+1} - \mathrm{z},\tau) \geq \mathcal{F}_{\mathcal{N}}(\mathrm{p}_n - \mathrm{p}^\star,\tau) + \tilde{\phi}(\mathfrak{B}(\mathrm{p}_n,\mathrm{p}^\star,\mathrm{p}_{n+1},z,\tau)).$$

On other hand,

$$\mathfrak{B}(\mathrm{p}_n,\mathrm{p}^\star,\mathrm{p}_{n+1},z,\tau)=\min\{\mathcal{F}_{\mathcal{N}}(\mathrm{p}_n - \mathrm{p}^\star,\tau), \max\{\mathcal{F}_{\mathcal{N}}(\mathrm{p}_n - \mathrm{p}_{n+1},\tau), \mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - z,\tau)\}\}$$

Letting $n \to \infty$ then :

$$\lim_{n\to\infty} \mathfrak{B}(\mathrm{p}_n,\mathrm{p}^\star,\mathrm{p}_{n+1},z,\tau) = 1$$

Hence $\mathfrak{B}(\mathrm{p}_n,\mathrm{p}^\star,\mathrm{p}_{n+1},z,\tau) = 1$ as $n \to \infty$

Thus $\mathcal{F}_{\mathcal{N}}(\mathrm{p}_{n+1} - \mathrm{z},\tau) \geq \mathcal{F}_{\mathcal{N}}(\mathrm{p}_n - \mathrm{p}^\star,\tau) + \tilde{\phi}(\mathfrak{B}(\mathrm{p}_n,\mathrm{p}^\star,\mathrm{p}_{n+1},z,\tau))$.

In the previous inequality, if take the limit as $n \to \infty$, then

$$\mathcal{F}_{\mathcal{N}}(p^\star - z,\tau) \geq 1 + \tilde{\phi}(1) \geq 1.$$

Thus $\mathcal{F}_{\mathcal{N}}(p^\star - z,\tau) = 1$ for each $\tau > 0$. Therefore $\mathrm{p}^\star = \mathrm{z}$ and $\mathcal{F}_{\mathcal{N}}(p^\star - \mathbb{T}p^\star,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$.

Eventually, to prove that $\mathrm{p}^\star$ is the unique $\mathcal{BPP}$ of $\mathbb{T}$. Consider $\omega \neq \mathrm{p}^\star$ other $\mathcal{BPP}$ of $\mathbb{T}$, that is, $\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \mathbb{T}\mathrm{p}^\star,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$ and $\mathcal{F}_{\mathcal{N}}(\omega - \mathbb{T}\omega,\tau) = N_d(\widetilde{U},\widetilde{V},\tau)$. If condition (e) of the theorem holds, then from inequality 9,

$$\mathcal{F}_{\mathcal{N}}(p^\star - \omega,\tau) \geq \mathcal{F}_{\mathcal{N}}(p^\star - \omega,\tau) + \tilde{\phi}(\mathfrak{B}(\mathrm{p}^\star,\omega,\mathrm{p}^\star,\omega,\tau)).$$

where

$$\mathfrak{B}(\mathrm{p}^\star,\omega,\mathrm{p}^\star,\omega,\tau) = \min\{\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \omega,\tau), \max\{\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \mathrm{p}^\star,\tau), \mathcal{F}_{\mathcal{N}}(\omega - \omega,\tau)\}\} = \mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \omega,\tau).$$

Therefore, $\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \omega,\tau) \geq \mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \omega,\tau) + \tilde{\phi}(\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \omega,\tau))$ and so $\tilde{\phi}\big(\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \omega,\tau)\big) = 0$, but this contradiction, therefore $\mathcal{F}_{\mathcal{N}}(\mathrm{p}^\star - \omega,\tau) = 1$ for each $\tau > 0$ and so $\mathrm{p}^\star = \omega$.

Example 2: Consider $L = \mathbb{R}$ with the fuzzy norm, $\mathcal{F}_{\mathcal{N}} : L \times \mathbb{R} \to [0,1]$ specified by:

$\mathcal{F}_{\mathcal{N}}(\mathrm{p},\tau) = \frac{\tau}{\tau+\|\mathrm{p}\|}$, for each $\mathrm{p} \in L$, $\tau > 0$ where $\|\mathrm{p}\| = |\mathrm{p}|$.
Let $\widetilde{\mathrm{U}} = \{2,3,4\}$ and $\widetilde{\mathrm{V}} = \{6,7,8,9,10\}$. Define $\mathbb{T} : \widetilde{U} \to \widetilde{V}$ by

$$\mathbb{T}\mathrm{p} = \begin{cases} 6 & if\ \mathrm{p} = 4 \\ \mathrm{p} + 4, & otherwise \end{cases}$$

and the mapping $\tilde{\alpha} : \widetilde{U} \times \widetilde{U} \times (0,\infty) \to [0,+\infty)$ given by:

$\tilde{\alpha}(\mathrm{p},\mathrm{q},\tau) = 1$ for each $\mathrm{p},\mathrm{q} \in \widetilde{U}$

Clearly, $\mathrm{N_d}(\widetilde{\mathrm{U}},\widetilde{\mathrm{V}},\tau) = \sup\{\mathcal{F}_{\mathcal{N}}(\mathrm{p}-\mathrm{q},\tau) : \mathrm{p} \in \widetilde{\mathrm{U}}, \mathrm{q} \in \widetilde{\mathrm{V}}\} = \frac{\tau}{\tau+2}$
Thus,
$\widetilde{U}_\circ(\tau) = \{\mathrm{p} \in \widetilde{U} : \mathcal{F}_{\mathcal{N}}(\mathrm{p}-\mathrm{q},\tau) = \frac{\tau}{\tau+2}$ for some $\mathrm{q} \in \widetilde{V}\} = \{4\}$
$\widetilde{V}_\circ(\tau) = \{\mathrm{q} \in \widetilde{V} : \mathcal{F}_{\mathcal{N}}(\mathrm{p}-\mathrm{q},\tau) = \frac{\tau}{\tau+2}$ for some $\mathrm{p} \in \widetilde{U}\} = \{6\}$.
It's clear that $T(\widetilde{U}_\circ(\tau)) \subseteq \widetilde{V}_\circ(\tau)$.

Suppose that

$$\begin{cases} \tilde{\alpha}(\mathrm{p},\mathrm{q},\tau) \leq 1 \\ \mathcal{F}_{\mathcal{N}}(\mathcal{u} - \mathbb{T}p,\tau) = N_d(\widetilde{U},\widetilde{V},\tau) \\ \mathcal{F}_{\mathcal{N}}(\mathcal{v} - \mathbb{T}q,\tau) = N_d(\widetilde{U},\widetilde{V},\tau) \end{cases}$$

Then

$$\begin{cases} \mathrm{p},\mathrm{q} \in \widetilde{U} \\ \mathcal{F}_{\mathcal{N}}(\mathcal{u} - \mathbb{T}p,\tau) = N_d(\widetilde{U},\widetilde{V},\tau) \\ \mathcal{F}_{\mathcal{N}}(\mathcal{v} - \mathbb{T}q,\tau) = N_d(\widetilde{U},\widetilde{V},\tau) \end{cases}$$

Hence $\mathcal{u} = \mathcal{v} = 4$ that is $\tilde{\alpha}(\mathcal{u},\mathcal{v},\tau) \leq 1$, which means $\mathbb{T}$ is an $\tilde{\alpha}-$ proximal admissible mapping.

Additional,

$$\mathcal{F}_{\mathcal{N}}(\mathcal{u} - \mathcal{v},\tau) = \frac{\tau}{\tau + |0|} = 1 \geq \mathcal{F}_{\mathcal{N}}(\mathrm{p}-\mathrm{q},\tau) + \tilde{\phi}\big(\mathfrak{B}(\mathrm{p}\ \mathrm{q},\mathcal{u},\mathcal{v},\tau)\big)$$

so $\mathbb{T}$ is an $\tilde{\alpha}$–$\tilde{\phi}$ proximal contractive mapping with $\tilde{\phi} : [0,1] \to [0,1]$ defined by $\tilde{\phi}(\mu) = 1 - \mu$ for each $\mu \in [0,1]$. Hence each condition of Theorem 2 holds and $\mathbb{T}$ possesses a unique $\mathcal{BPP}$. $\mathrm{p}^\star = 4$ represents $\mathcal{BPP}$ of $\mathbb{T}$ in this example.

Conclusions:

In this paper, the concept of $\tilde{\alpha}-\tilde{\psi}$ proximal contractive and $\tilde{\alpha}-\tilde{\phi}$ proximal contractive mappings in a $\mathcal{F}_{\mathcal{N}}$ space is presented and the best proximity point theorem for these types of mappings in a $\mathcal{F}_{\mathcal{N}}$ space is proved. To show the usefulness of the produced results, certain examples are offered. In future work, more research is needed on the generalizations of these types of contraction mappings and study the applications for these mappings in the fuzzy normed space.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at the University of Baghdad.

Authors' contributions statement:

R. I. S. contributed to the research design and implementation, as well as the analysis of the results. B. A. A. participated in the revision of the results and proofreading. Both authors worked together to enhance the final research by discussing the results.

References:

1. Zadeh L. Fuzzy sets. Ann Infect Control. 1965; 8(3): 338-353.
2. Kramosil I, Michalek J. Fuzzy metrics and statistical metric spaces. Kybernetika. 1975; 11(5): 336–344.
3. George A, Veeramani P.On some results in fuzzy metric spaces. Fuzzy Sets Syst. 1994; 64(3): 395–399.
4. Gregori V, Minana J, Morillas S. On completable fuzzy metric spaces. Fuzzy Sets Syst. 2015; 267(15): 133-139.
5. Zainab A, Kider JR. The Hausdorff Algebra Fuzzy Distance and its Basic Properties. Eng Technol J. 2021; 39(7): 1185-1194.
6. Sabri RI. Compactness Property of Fuzzy Soft Metric Space and Fuzzy Soft Continuous Function. Iraqi J Sci. 2021; 62(9): 3031–3038.
7. Paknazar M. Non-Archimedean fuzzy metric spaces and best proximity point theorems. Sahand Commun Math Anal. 2018; 9(1): 85-112.
8. Katsaras A. Fuzzy topological vector spaces II. Fuzzy Sets Syst. 1984; 12(2): 143–154.
9. Kider JR, Jameel RM. Fuzzy Bounded and Continuous Linear Operators on Standard Fuzzy Normed Spaces. Eng Technol J. 2015; 33(2): 178–185.
10. Kider JR, Noor AK. Properties of Fuzzy Closed Linear Operator. Eng Technol J. 2019; 37(18): 25-31.
11. Gheeab MN, Kider JR. Properties of the Adjoint Operator of a General Fuzzy Bounded Operator. Baghdad Sci J. 2021; 18(1).
12. Sabri RI. Fuzzy Convergence Sequence and Fuzzy Compact Operators on Standard Fuzzy Normed Spaces. Baghdad Sci J. 2021; 18(4): 1204-1211.
13. Fan K. Extensions of two fixed point theorems of F. E. Browder. Math Z 112. 1969;7: 234–240.
14. Gabriela IU, Teodor T. Best proximity points of (EP)-operators with qualitative analysis and simulation. Math Comput Simul. 2021; 2021(187): 215-230.
15. Paknazar M, Sen D. Some new approaches to modular and fuzzy metric and related best proximity results. Fuzzy Sets Syst. 2020; 390(1): 138-159.
16. Ishak A, Hakan S, Mustafa A. A new approach to fractals via best proximity point. Chaos Solit. Fractals. 2021; (146): 1-7.
17. Iqbal I, Hussain N, Kutbi MA. Existence of the solution to variational inequality, optimization problem, and elliptic boundary value problem through revisited best proximity point results. J Comput Appl Math. 2020; 375: 112804.
18. Nadaban S, Dzitac I. Atomic decompositions of fuzzy normed linear spaces for wavelet applications. Informatica. 2014; 25(4): 643–662.
19. Bag T, Samanta SK. Finite-dimensional fuzzy normed linear spaces. Ann. fuzzy math. inform. 2003; 6(2): 271–283.
20. Vetro C, Salimi P. Best Proximity Point Results in Non-Archimedean Fuzzy Metric Spaces. Fuzzy Inf Eng. 2013; 5(4): 417-429.
21. Saha P, Guria S, Binayak S. Determining Fuzzy Distance Through Non-Self Fuzzy Contractions. Yugosl J Oper Res. 2019; 29(3): 325-335.

أفضل نظرية نقطة تقارب لدالة الانكماش $\tilde{\Psi}-\tilde{\alpha}$ في الفضاء المعياري الضبابي

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الخلاصة:

أفضل نقطة تقارب هي تعميم النقطة الثابتةحيث يكون مفيدًا عندما لا تكون دالة الانكماش دالة ذاتية ، من ناحية أخرى ، توفر أفضل نظريات التقريب حلاً تقريبيًا لمعادلة النقطة الثابتة $\mathbb{T}p = p$ وهي تستخدم لحل المسألة من أجل التوصل الى تقريب جيد. الهدف الرئيسي من هذا البحث هو تقديم أنواع جديدة من الانكماش القريب للدوال غير الذاتية في الفضاء المعياري الضبابي ثم إثبات نظرية نقطة التقارب الأفضل لهذه الدوال. في البداية ، يتم تقديم تعريف الفضاء المعياري الضبابي. ثم يتم تقديم مفاهيم أفضل نقطة تقارب و α - القريبة المقبولة في سياق الفضاء المعياري الضبابي. نقدم مفهوم الدالة الانكماشية القريبة $\tilde{\psi}-\tilde{\alpha}$. بعد ذلك ، نظرية أفضل نقطة تقارب لمثل هذا النوع من الدوال في الفضاء المعياري الضبابي تم برهانها. بالإضافة إلى ذلك ، يتم تقديم مفهوم الدالة الانكماشية القريبة $\tilde{\phi}-\tilde{\alpha}$ في الفضاء المعياري الضبابي وفي ظل ظروف محددة ، تم برهان نظرية أفضل نقطة تقارب لمثل هذا النوع من الدوال ، علاوة على ذلك ، تم تقديم بعض الأمثلة لإثبات إمكانية تطبيق النتائج.

الكلمات المفتاحية: أفضل نقطة تقارب، فضاء معياري ضبابي، الدالة المقبولة القريبه $\tilde{\alpha}$-، دالة الانكماش القريبة $\tilde{\Psi}-\tilde{\alpha}$، دالة الانكماش القريبة $\tilde{\phi}-\tilde{\alpha}$.