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A Characterization of Maximal Outerplanar-Open Distance Pattern Uniform Graphs

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Abstract:

Let A ⊆ V(H) of any graph H, every node w of H be labeled using a set of numbers; $foA(w) = \{d(w,v): v \in A, w \neq v\}$, where d(w,v) denotes the distance between node w and the node v in H, known as its open A-distance pattern. A graph H is known as the open distance-pattern uniform (odpu)-graph, if there is a nonempty subset A ⊆V(H) together with $foA(s)$ is the same for all $s \in V(H)$. Here foA is known as the open distance pattern uniform (odpu-) labeling of the graph H and A is known as an odpu-set of H. The minimum cardinality of vertices in any odpu-set of H, if it exists, will be known as the odpu-number of the graph H. This article gives a characterization of maximal outerplanar-odpu graphs. Also, it establishes that the possible odpu-number of an odpu-maximal outerplanar graph is either two or five only.

Keywords: Central Subgraphs, Maximal Outerplanar Graphs, ODPU- Graphs, ODPU-Number, ODPU- Set.

Introduction:

Every graph in this article is simple as well as connected. A nonempty subset A of nodes of any graph G, an A-distance pattern of any node s of G has been defined as a set f^o_A (s) = {d(s, v): v ∈ A}, where d(s,v) denote the distance between node s and the node v in G; it is clear that, 0 ∈f_A(u) iff d(u, v) = 0 for some v ∈ A iff u = v and u ∈ A. This observation motivated associating with each vertex u of G its open A-distance pattern (or, A-`odp') f^o_A(s) = {d(s, t): t ∈ A, s ≠ t}. The problem is to find those graphs G that has a nonempty set A ⊆ V(G) together with f^o_A (u) is the same for every vertices u; then, denote such a graph the odp-uniform graphs, where f^o_A is known as the open distance pattern uniform (or, a odpu-) labeling and set A is known as odpu-set in G. Also, if exists, the minimum cardinality of the nodes of an odpu-set of G is known as the odpu-number of G.

It is proved that a graph G with radius r(G) is an odpu graph if and only if the open distance pattern of every vertex in G is {1, 2, …, r(G)} and proved that a graph is an odpu-graph if and only if its center Z (G) is an odpu-set, thereby characterizing odpu-graphs, which in fact invokes a method to check the existence of an odpu-set for any given graph[1].

In this article, it is studied maximal outerplanar-odpu graphs. It is given a characterization of maximal outerplanar-odpu graphs. Also, it is found that the possible odpu-number of an odpu-maximal outerplanar graph is either two or five only.

Currently, existing definitions and results are going to use in this article.

Theorem1:[1]For any graph G, odpu number of G is 2 if and only if there exist at least two vertices x, y ∈ V(G) such that d(x) = d(y) = |V(G)|-1.

Theorem2:[1]A graph G is an odpu graph if and only if its center Z(G) is an odpu set and hence |Z(G)| ≥ 2.

Theorem 3:[1] All self-centered graphs are odpu graphs.

Theorem4:[1]Every odpu-graph G satisfies, r(G) ≤ d(G) ≤ r(G)+1 where r(G) and d(G) denote the radius and diameter of G respectively.

Chordal graphs are well-studied in the article[1]. A chordal graph is a graph G, whose each cycle with a minimum length of four has one chord. That is, a graph is chordal if each non-consecutive node of any cycle is made adjacent by an edge in that graph[2]. Similar distance-related metric-related domination concepts were found in articles[3,4]and other types of metric dimension concepts can be found in articles[5-7].

Proposition 1:[1]A chordal graph G is an odpu-graph then ⟨Z(G)⟩ is self-centered and since r(G) = r(⟨Z(G)⟩), ⟨Z(G)⟩ is also self-centered.

Maximal Outerplanar graphs are well studied in the article[2]. A graph is said to be outerplanar, if that may be able to be drawn in a plane having all vertices lie in the exterior boundary of that graph. Any graph is called maximal outerplanar, if the addition of any one of the edges made the graph non-outerplanar. Every maximal outerplanar graph is chordal[2].

Proposition 2:[2]If G is a maximal outerplanar graph, then its central subgraph⟨Z(G)⟩ is isomorphic to one of the seven graphs in Fig.1.

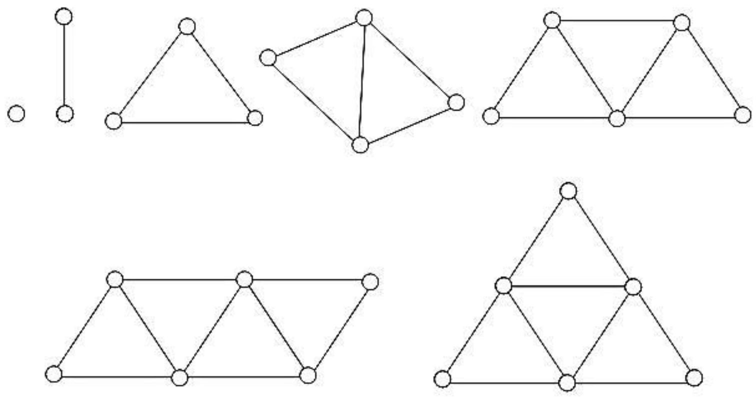

Figure 1. Central Subgraph of Maximal Outerplanar Graphs

Maximal Outerplanar-odpu-Graphs:

The next theorem establishes one necessary condition for any maximal outerplanar graph to become an odpu-graph. The condition is based on some specific structure of the induced subgraph of the center, ⟨Z(G)⟩ of the maximal outerplanar graphs.

Theorem 5: The induced subgraph of the center, ⟨Z(G)⟩ of any maximal outer-planar-odpu-graph G will be isomorphic to any graphs given in Fig.2.

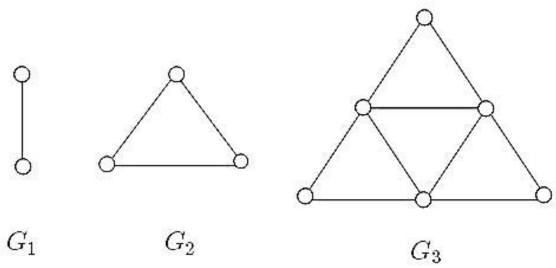

Figure 2. Induced Central Subgraph of Maximal Outerplanar ODPU Graphs

Proof: Proposition 2, ⟨Z(G)⟩ gives that any maximal outerplanar graph will be isomorphic to any of the graphs listed in Fig.1.

It is already proved that all maximal outer-planar graphs are chordal (cf. [2]). By Proposition 1, given above, ⟨Z(G)⟩ be self-centered. Hence it can be easily verified that ⟨Z(G)⟩ will be isomorphic to any graphs listed in Fig.2. This proves the result.

Theorem 6: Let G, be a maximal outer-planar graph. Then G can be odpu iff G will be isomorphic to any graphs listed in Fig.3.

Proof: Assume that graph G is maximal outer-planar which is odpu. Then Theorem 5 gives that ⟨Z(G)⟩ will be isomorphic to any one graph listed in the figure-2. Hence by Proposition 1, the radius of G and radius of ⟨Z(G)⟩isthe same. Since the radius is either 1 or 2, there are only two cases. In the first case is both radius is one and in the second, both two.

Case-(i)*both radius is equal to one.* Then by Theorem 1, there exists a minimum of 2 vertices in G, which are universal (i.e., whose degree is one less than the number of vertices of it). Hence the subgraph⟨Z(G)⟩ will be isomorphic to any one of G_1 and G_2 listed in the figure-2. Now, take every graph, which has a minimum of two universal vertices. It is clear that P_2 is the smallest among these classes of graphs. Suppose P_2 is isomorphic to H_1 is an edge xy. In the next steps, try to add nodes by getting another maximal outerplanar graph, without changing the universal degree status of the nodes x and y in the graph G. ie, G = xy+{w_1, w_2, …, w_k}, and the '+' is the graph operator denoted by join (The join of two graphs $G_1 = (V_1, E)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1 + G_2$ has the vertex set as $V= V_1 \cup V_2$ and the edge set E contains all the edges of G_1 and G_2 together with all edges joining the vertices of V_1 with the vertices of V_2) of the 2 graphs. The case i = 1 implies that G = xy + w_1 is a cycle C_3 that is the graph H_2, given in the list. Clearly, it is maximal outer-planar-odpu. The case i=2 implies that G = xy + {w_1, w_2} is isomorphic to the graph H_3 given in the list. Clearly, it also is maximal outer-planar-odpu. Finally the case i ≥ 3 implies that, G = xy + {w_1, w_2, …, i} is isomorphic to K_2 + Complement of (K_j), j ≥ 3. They cannot be maximal outer-planar. Hence, the only graphs that come under the category of maximal outer-planar graphs satisfying the condition of both radius of G and r(⟨Z(G)⟩) are one is H_1, H_2, and H_3.

Case-(ii) *both radiuses are equal to two.* In this case, Theorem 5, clearly establishes that ⟨Z(G)⟩ will be isomorphic to the graph G_3 which is the same as H_4. But H_4 is a self-centered graph. So H_4 is odpu. If, there is an odpu-graph H which is maximal outer-planar and not isomorphic to H_4, having the property ⟨Z(H)⟩ which will be isomorphic to the graph G_3, implies that there is a node w, not in V(H)-Z(H) with w is made adjacent to at least one node an of ⟨Z(H)⟩. Maximal outer-planarity of H implies that the vertex w is not made adjacent with each node b, c, and a'. Suppose t is a vertex with (wta') will be one path H. Thus S_0 (a, a', H) is not a subset of Z(H) = G_3, which is a contradiction. Thus, d(w, a') = 3. Hence the eccentricity e_H(a') is greater than there and so a' not in Z(H), is also a contradiction. So, such a vertex x does not exist which is other than the vertices of G_3. Thus, in this case, the only graph which is maximal outerplanar-odpu satisfying both radius is two is H_4.

Theorem 7: The possible odpu-number of an odpu-maximal outerplanar graph is either two or five only.

Proof: Suppose G be maximal outer-planar odpu. Then, from Theorem 6, G will be isomorphic to any one graph listed in Fig.3.

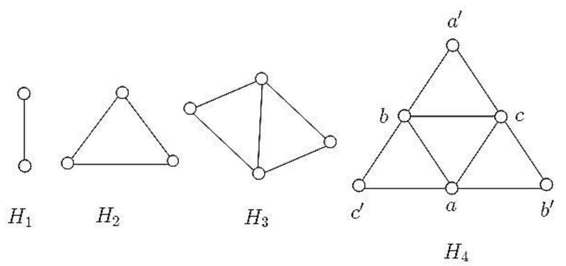

Figure 3. Maximal Outerplanar ODPU Graphs

H_1, H_2, and H_3 have two universal vertices and hence the odpu-number of H_1, H_2, and $H_3$are 2. Now, H_4 is a self-centered graph having a radius of two. So for any odpu-set M, $f^o{}_M$(u) = {1, 2}, for every vertex u ∈V(H_4). Suppose the minimal odpu-set of H_4 is M. Obviously, there is no vertex other than a, at a distance two from a' ∈V(H_4)and so a' belongs to M. In the same manner, the nodes b' and c' will be the corresponding nodes in H_4 having a distance 2 from nodes b and c correspondingly. So nodes b', c belongs to M. So M contains more than three nodes. Suppose that M = {a', b', c'}. In this case, 1 is not in $f^o{}_M$ (v), where v = a', b', c'. Hence at least, a ∈ M. So the set M must contain the set {a, a', b', c'}. In this case also, if consider the vertex a, 1 is not in $f^o{}_M$ (a'), and so any one node b or c belongs to M. Thus assume, b is in M. So let M = {a, b, a', b', c'}. Thus, $f^o{}_M$ (v) = {1, 2} for every v ∈V(H_4). Hence odpu number of H is five. So the possible odpu-number of an odpu-maximal outerplanar graph will be either two or five only. This completes the proof.

Conclusion:

Characterization of a few classes of odpu graphs has been done, including maximal outer planar odpu graphs. But the general characterization of odpu graphs is still an open problem.

Authors' declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Sanatana Dharma College Alappuzha.

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توصيف المخطط الخارجي الأقصى- نمط المسافة المفتوحة رسومات موحدة

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الخلاصة:

ليكن $A \subseteq V(H)$ لأي رسم بياني H ، كل عقدة w من H يتم تصنيفها باستخدام مجموعة من الأرقام ; $foA\ (w) = \{d\ (w, v): v \in A, w \neq v\}$ ، حيث تشير d (w، v) إلى المسافة بين العقدة w والعقدة v في H ، والمعروفة باسم نمط المسافة A المفتوح . يُعرف الرسم البياني H بأنه الرسم البياني لنمط المسافة المفتوحة (odpu) - الرسم البياني ، إذا كانت هناك مجموعة فرعية غير فارغة $A \subseteq V(H)$ مع $foA\ (s)$ هي نفسها لجميع $s \in V(H)$. هنا تُعرف foA باسم النمط الموحد لنمط المسافة المفتوحة (odpu-) للرسم البياني H و A يُعرف بمجموعة- odpu من H. سوف يعرف الحد الادنى من رؤوس الكاردينال لاي مجموعة-odpu من H ان وجد كعدد –odpu للرسم البياني H . في هذه المقالة ، نعطي توصيفًا للرسوم البيانية الخارجية القصوى للمخطط -odpu . كما وجدنا أيضًا أن الرقم الفردي المحتمل للرسم البياني اما يكون اثنان او خمسة فقط.

الكلمات المفتاحية: الرسم البياني الجزئي المركزي، الرسم البياني الأقصى للخطة، الرسوم البيانية - ODPU ، رقم - ODPU، مجموعة-ODPU.