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Fractional Hartley Transform and its Inverse

Vasant Gaikwad*

Department of Mathematics, Shri. Chhatrapatti Shivaji College, Omerga, Osmanabad, Maharastra, Pin Code – 413606, India.
*Corresponding author: gaikwadvasant19@gmail.com
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Abstract:

The Hartley transform generalizes to the fractional Hartley transform (FRHT) which gives various uses in different fields of image encryption. Unfortunately, the available literature of fractional Hartley transform is unable to provide its inversion theorem. So accordingly original function cannot retrieve directly, which restrict its applications. The intension of this paper is to propose inversion theorem of fractional Hartley transform to overcome this drawback. Moreover, some properties of fractional Hartley transform are discussed in this paper.

Keywords: Fourier transform, Fractional Fourier transform, Fractional Hartley transform, Hartley transform, Inverse.

Introduction:

In the early decades fractional Fourier transform comes into the picture very numerously because of its huge applications in the field of optical propagation problems[1], n- dimensional FRFT [2] and optical image encryption algorithm[3] etc.

As fractional Hartley transform (FRHT) is closely related with fractional Fourier transform (FRFT), for each application of FRFT there is one application of FRHT and as Hartley transform is better than Fourier transform so FRHT has benefits over FRFT.

A general theory of linear fractional transform has come up with a symmetric procedure in order to define the fractional version of any established linear transform[4]. The definition of fractional Fourier transform and fractional Hartley transform which satisfy the boundary conditions and additive property simultaneously are derived continuously by using general theory of linear fractional transform[4]. Many applications of fractional Hartley transform could be found collaterally just like that of fractional Fourier transform.

It is seen that, FRHT and other transforms has several applications in the area of G-Boehmian space[5], cryptosystem[6], asymmetric watermarking scheme[7], asymmetric encryption algorithm for colour images[8], multi-image encryption[9-11] and image steganograpic[12]. Due to many applications of FRFT and FRHT, an attempt has been made to write this paper on FRHT and its inversion theorem. Moreover, it contains some properties of FRHT and its kernel.

Definition. 1:[4] If $\alpha \in \mathbb{R}$; with α is a constant and $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, then FRFT is denoted by $R^\alpha[h(t)](v)$ or $F_\alpha(v)$ or $g_F^\alpha(v)$ and is defined by

$$R^\alpha[h(t)](v) = F_\alpha(v) = g_F^\alpha(v) = \int_{-\infty}^{\infty} h(t) K_F^\alpha(t, v)\, dt, \text{ where}$$

$$K_F^\alpha(t, v) = \sqrt{\frac{1 - i\cot\psi}{2\pi}} \exp\left\{ i\left[\frac{1}{2}(t^2 + v^2)\cot\psi - tv\ cosec\,\psi\right]\right\}$$ and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$

Theorem. 1:[2] If $\alpha, t, v \in \mathbb{R}$, where α is constant, $R^\alpha[h(t)](v) = F_\alpha(v) = g_F^\alpha(v)$ is the FRFT of h(t), then $h(t)$ is as follows

$$h(t) = \int_{-\infty}^{\infty} \overline{K_F^\alpha(t, v)}\, g_F^\alpha(v), \text{ where}$$

$$\overline{K_F^\alpha(t, v)} = \sqrt{\frac{1 + i\cot\psi}{2\pi}} \exp\left\{ -i\left[\frac{1}{2}(t^2 + v^2)\cot\psi - tv\ cosec\,\psi\right]\right\}$$ and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$

Property. 1:[1] Shifting Property
The shifting property for FRFT is

$$R^{\alpha}[h(t+y)](v) = g_F^{\alpha}(v + y\cos\psi) \times \exp\left[\frac{1}{2} i y^2 \sin\psi\cos\psi + ivy\sin\psi\right].$$

Similarly, the shifting property for FRFT of negative variables is

$$R^{\alpha}[h(t+y)](-v) = g_F^{\alpha}(-v + y\cos\psi) \times \exp\left[\frac{1}{2} i y^2 \sin\psi\cos\psi + i(-v)y\sin\psi\right].$$

Where $\alpha,\ t,\ y,\ v \in \mathbb{R}$ and $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, with α and y are constants.

Definition. 2:[4] If $\alpha \in \mathbb{R},\ h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, then the FRHT of h(t) is denoted by $g_H^{\alpha}[h(t)](v)$ or $g_H^{\alpha}(v)$ and is defined as

$$g_H^{\alpha}[h(t)](v) = g_H^{\alpha}(v) = \int_{-\infty}^{\infty} K_H^{\alpha}(t,v) h(t) dt, \quad \ldots\ldots\ldots 1$$

where

$$K_H^{\alpha}(t,v) = \sqrt{\frac{1-i\cot\psi}{2\pi}}\ e^{i\frac{1}{2}(t^2+v^2)\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{i(\psi-\frac{\pi}{2})}\sin(tv\ cosec\,\psi)\right]$$

and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$

Note. 1: If $\psi = \frac{\pi}{2}$, then the extended transform defined in Definition 2 reduces to Hartley transform.

Result. 1:[4] The correlation between kernel of FRHT and kernel of FRFT is written by

$$K_H^{\alpha}(t,v) = \left[\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^{\alpha}(t,v) + \left[\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^{\alpha}(t,-v).$$

Main Results:

Result 2: If a kernel of FRFT is denoted by $K_F^{\alpha}(t,v)$ and a kernel of FRHT is denoted by $K_H^{\alpha}(t,v)$, then

$$K_F^{\alpha}(t,v) = \left[\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^{\alpha}(t,v) + \left[\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^{\alpha}(t,-v).$$

The Proof is obvious by Result 1.

Result. 3: If $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, the FRFT of $h(t)$ is denoted by $g_F^{\alpha}(v)$ and FRHT is denoted by $g_H^{\alpha}(v)$, then

$$g_H^{\alpha}(v) = \left[\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right] g_F^{\alpha}(v) + \left[\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right] g_F^{\alpha}(-v).$$

Proof: By Definition 2,

$$g_H^{\alpha}(v) = \int_{-\infty}^{\infty} h(t) K_H^{\alpha}(t,v) dt.$$

By Result 1 and Definition 1,

$$g_H^{\alpha}(v) = \int_{-\infty}^{\infty} h(t)\left\{\left[\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^{\alpha}(t,v) + \left[\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^{\alpha}(t,-v)\right\} dt$$

$$= \int_{-\infty}^{\infty} h(t)\left[\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^{\alpha}(t,v) dt + \int_{-\infty}^{\infty} h(t)\left[\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right] K_F^{\alpha}(t,-v)\, dt$$

$$= \left[\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right] g_F^{\alpha}(v) + \left[\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right] g_F^{\alpha}(-v).$$

Result. 4: If $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, the FRFT of $h(t)$ is denoted by $g_F^{\alpha}(v)$ and FRHT is denoted by $g_H^{\alpha}(v)$ then

$$g_F^{\alpha}(v) = \left[\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right] g_H^{\alpha}(v) + \left[\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right] g_H^{\alpha}(-v).$$

Proof: By Definition 1,

$$g_F^{\alpha}(v) = \int_{-\infty}^{\infty} h(t) K_F^{\alpha}(t,v) dt$$

By Result 2 and Definition 2,

$$g_F^{\alpha}(v) = \int_{-\infty}^{\infty} h(t)\left\{\left[\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^{\alpha}(t,v) + \left[\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^{\alpha}(t,-v)\right\} dt$$

$$= \int_{-\infty}^{\infty} h(t)\left[\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^{\alpha}(t,v) dt + \int_{-\infty}^{\infty} h(t)\left[\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right] K_H^{\alpha}(t,-v) dt$$

$$= \left[\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right] g_H^{\alpha}(v) + \left[\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right] g_H^{\alpha}(-v).$$

Theorem. 2: Inverse fractional Hartley transform. If $h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$; $t,\ v,\ \alpha \in \mathbb{R}^n$, where α is a constant, $g_H^{\alpha}(v)$ is the FRHT of h(t), then $h(t)$ is as follows

$$h(t) = \int_{-\infty}^{\infty} \overline{K_H^{\alpha}(t,v)}\, g_H^{\alpha}(v) dv,$$

where

$$\overline{K_H^{\alpha}(t,v)} = \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi} \times \left[\cos(tv\ cosec\,\psi) + e^{-i\left(\psi-\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right]$$

and $\psi = \frac{\alpha\pi}{2}$ if $\psi \neq \pi n$; for all $n = 0, 1, 2, \ldots$

Proof: By inversion formula of the FRFT and Result 4,

$$h(t) = \int_{-\infty}^{\infty} \overline{K_H^{\alpha}(t,v)}\, g_F^{\alpha}(v)dv$$
$$= \int_{-\infty}^{\infty} \overline{K_F^{\alpha}(t,v)}\left[\left(\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^{\alpha}(v) + \left(\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^{\alpha}(v)\right] dv$$
$$= \int_{-\infty}^{\infty} \overline{K_F^{\alpha}(t,v)}\left(\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^{\alpha}(v)dv + \int_{-\infty}^{\infty} \overline{K_F^{\alpha}(t,v)}\left(\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^{\alpha}(-v)dv$$
$$= \int_{-\infty}^{\infty} \overline{K_F^{\alpha}(t,v)}\left(\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^{\alpha}(v)dv + \int_{-\infty}^{\infty} \overline{K_H^{\alpha}(t,-v)}\left(\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right) g_H^{\alpha}(v)dv.$$

That is $h(t) = \int_{-\infty}^{\infty}\left[\overline{K_F^{\alpha}(t,v)}\left(\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right) + \overline{K_F^{\alpha}(t,-v)}\left(\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right)\right] g_H^{\alpha}(v)dv.$

Consider

$$\overline{K_F^{\alpha}(t,v)}\left(\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right) + \overline{K_F^{\alpha}(t,-v)}\left(\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right)$$
$$= \left(\frac{1+e^{\frac{-i\alpha\pi}{2}}}{2}\right)\sqrt{\frac{1+i\cot\psi}{2}} \times \exp\left\{-i\left[\frac{1}{2}(t^2+v^2)\cot\psi - tv\ cosec\,\psi\right]\right\}$$
$$+\left(\frac{1-e^{\frac{-i\alpha\pi}{2}}}{2}\right)\sqrt{\frac{1+i\cot\psi}{2\pi}} \times \exp\left\{-i\left[\frac{1}{2}(t^2+v^2)\cot\psi + tv\ cosec\,\psi\right]\right\}$$

$$= \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\left(\frac{e^{i\,tv\ cosec\,\psi}+e^{-i\,tv\ cosec\,\psi}}{2}\right) + ie^{\frac{-i\alpha\pi}{2}}\left(\frac{e^{i\,tv\ cosec\,\psi}-e^{-i\,tv\ cosec\,\psi}}{2i}\right)\right]$$

$$= \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{\frac{i\pi}{2}} e^{-i\psi}\sin(tv\ cosec\,\psi)\right]$$
$$= \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{-i\left(\psi-\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right].$$

Therefore

$$h(t) = \int_{-\infty}^{\infty}\sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{-i\left(\psi-\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right] g_H^{\alpha}(v)dv.$$

Hence $h(t) = \int_{-\infty}^{\infty}\overline{K_H^{\alpha}(t,v)}\, g_H^{\alpha}(v)dv$, where

$$\overline{K_H^{\alpha}(t,v)} = \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{-i\left(\psi-\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right]$$

and $\psi = \frac{\alpha\pi}{2}$ if $\alpha \neq \pi n$; for all $n = 0, 1, 2, \ldots$

Property. 2: Symmetric Property.
$K_H^{\alpha}(t,v) = K_H^{\alpha}(v,t)$; $t,\ \alpha,\ v \in \mathbb{R}$ with α is a constant.

Property. 3: Conjugation Property.
$\overline{K_H^{\alpha}(t,v)} = K_H^{(-\alpha)}(t,v)$; where $\overline{K_H^{\alpha}(t,v)}$ is a conjugate of $K_H^{\alpha}(t,v)$.

Proof: The kernel of FRHT is,

$$K_H^{\alpha}(t,v) = \sqrt{\frac{1-i\cot\psi}{2\pi}}\, e^{i\frac{t^2+v^2}{2}\cot\psi} e^{i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{i\left(\psi-\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right],$$

$\psi = \frac{\alpha\pi}{2}$ if $\alpha \neq \pi n$; for all $n = 0, 1, 2, \ldots$ and

$$\overline{K_H^{\alpha}(t,v)} = \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{-i\left(\psi-\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right],$$

$\psi = \frac{\alpha\pi}{2}$ if $\alpha \neq \pi n$; for all $n = 0, 1, 2, \ldots$

Therefore

$$K_H^{(-\alpha)}(t,v) = \sqrt{\frac{1-i\cot\left(\frac{-\alpha\pi}{2}\right)}{2\pi}}\, e^{i\frac{t^2+v^2}{2}\cot\left(\frac{-\alpha\pi}{2}\right)}\left[\cos\left(tv\ cosec\left(\frac{-\alpha\pi}{2}\right)\right) + e^{i\left(\left(\frac{-\alpha\pi}{2}\right)-\frac{\pi}{2}\right)}\sin\left(tv\ cosec\left(\frac{-\alpha\pi}{2}\right)\right)\right]$$

$$= \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) - e^{-i\left(\psi+\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right]$$

$$= \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\,cosec\,\psi) + e^{i\pi} e^{-i\left(\psi+\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right]$$

$$= \sqrt{\frac{1+i\cot\psi}{2\pi}}\, e^{-i\frac{t^2+v^2}{2}\cot\psi}\left[\cos(tv\ cosec\,\psi) + e^{-i\left(\psi-\frac{\pi}{2}\right)}\sin(tv\ cosec\,\psi)\right]$$

$$= \overline{K_H^{\alpha}(t\,,v)}$$

Hence $\overline{K_H^{\alpha}(t\,,v)} = K_H^{(-\alpha)}(t\,,v)$.

Property.4:
$K_H^{\alpha+\beta}(t\,,v) = \int_{-\infty}^{\infty} K_H^{\alpha}(t\,,v')K_H^{\beta}(v',v)\ dv'$.
Proof: Consider
$\int_{-\infty}^{\infty} K_H^{\alpha}(t\,,v')K_H^{\beta}(v',v)\ dv' =$

$$\int_{-\infty}^{\infty}\left[\left(\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right)K_F^{\alpha}(t\,,v') + \left(\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right)K_F^{\alpha}(t\,,-v')\right]\left[\left(\frac{1+e^{\frac{i\beta\pi}{2}}}{2}\right)K_F^{\beta}(v',v) + \left(\frac{1-e^{\frac{i\beta\pi}{2}}}{2}\right)K_F^{\beta}(v',-v)\right]dv'$$

$$= \left(\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right)\left(\frac{1+e^{\frac{i\beta\pi}{2}}}{2}\right)\int_{-\infty}^{\infty} K_F^{\alpha}(t\,,v')K_F^{\beta}(v',v)dv' + \left(\frac{1+e^{\frac{i\alpha\pi}{2}}}{2}\right)\left(\frac{1-e^{\frac{i\beta\pi}{2}}}{2}\right)\int_{-\infty}^{\infty} K_F^{\alpha}(t\,,v')\,K_F^{\beta}(v',-v)dv'$$

$$+\left(\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right)\left(\frac{1+e^{\frac{i\beta\pi}{2}}}{2}\right)\int_{-\infty}^{\infty} K_F^{\alpha}(t\,,-v')K_F^{\beta}(v',v)dv' + \left(\frac{1-e^{\frac{i\alpha\pi}{2}}}{2}\right)\left(\frac{1-e^{\frac{i\beta\pi}{2}}}{2}\right)\int_{-\infty}^{\infty} K_F^{\alpha}(t\,,-v')K_F^{\beta}(v',v)\ dv'$$

$$= \frac{1+e^{\frac{i\alpha\pi}{2}}+e^{\frac{i\beta\pi}{2}}+e^{\frac{i\alpha\pi}{2}}e^{\frac{i\beta\pi}{2}}}{4}K_F^{\alpha+\beta}(t\,,v) + \frac{1+e^{\frac{i\alpha\pi}{2}}-e^{\frac{i\beta\pi}{2}}-e^{\frac{i\alpha\pi}{2}}e^{\frac{i\beta\pi}{2}}}{4}K_F^{\alpha+\beta}(t\,,-v)$$

$$+\frac{1-e^{\frac{i\alpha\pi}{2}}+e^{\frac{i\beta\pi}{2}}-e^{\frac{i\alpha\pi}{2}}e^{\frac{i\beta\pi}{2}}}{4}\int_{-\infty}^{\infty} K_F^{\alpha}(t\,,-v')K_F^{\beta}(-v',-v)dv'$$

$$+\frac{1-e^{\frac{i\alpha\pi}{2}}-e^{\frac{i\beta\pi}{2}}+e^{\frac{i\alpha\pi}{2}}e^{\frac{i\beta\pi}{2}}}{4}\int_{-\infty}^{\infty} K_F^{\alpha}(t\,,-v')K_F^{\beta}(-v',v)dv'$$

$$= \frac{1+e^{\frac{i\alpha\pi}{2}}+e^{\frac{i\beta\pi}{2}}+e^{\frac{i(\alpha+\beta)\pi}{2}}}{4}K_F^{\alpha+\beta}(t\,,v) + \frac{1+e^{\frac{i\alpha\pi}{2}}-e^{\frac{i\beta\pi}{2}}-e^{\frac{i(\alpha+\beta)\pi}{2}}}{4}K_F^{\alpha+\beta}(t\,,-v)$$

$$+\frac{1-e^{\frac{i\alpha\pi}{2}}+e^{\frac{i\beta\pi}{2}}-e^{\frac{i(\alpha+\beta)\pi}{2}}}{4}K_F^{\alpha+\beta}(t\,,-v) + \frac{1-e^{\frac{i\alpha\pi}{2}}-e^{\frac{i\beta\pi}{2}}+e^{\frac{i(\alpha+\beta)\pi}{2}}}{4}K_F^{\alpha+\beta}(t\,,v)$$

$$= \frac{\left(1+e^{i\frac{(\alpha+\beta)\pi}{2}}\right)}{2}K_F^{\alpha+\beta}(t\,,v) + \frac{\left(1-e^{i\frac{(\alpha+\beta)\pi}{2}}\right)}{2}K_F^{\alpha+\beta}(t,-v)$$

$= K_H^{\alpha+\beta}(t\,,v)$.

Property. 5: Linearity.
If $g_H^{\alpha}[h_1(t)](v), g_H^{\alpha}[h_2(t)](v)$ are the FRHT of $h_1(t), h_2(t)$ respectively, then FRHT of $\alpha_1 h_1(t) + \alpha_2 h_2(t)$ is given by
$g_H^{\alpha}[\alpha_1 h_1(t) + \alpha_2 h_2(t)](v)$
$= \alpha_1 g_H^{\alpha}[h_1(t)](v) + \alpha_2 g_H^{\alpha}[h_2(t)](v)$.

Property. 6: Commutativity.
$g_H^{\alpha+\beta}(v) = g_H^{\alpha}\circ g_H^{\beta}(v) = g_H^{\beta}\circ g_H^{\alpha}(v) = g_H^{\beta+\alpha}(v)$.
That is $g_H^{\alpha+\beta} = g_H^{\alpha}\circ g_H^{\beta} = g_H^{\beta}\circ g_H^{\alpha} = g_H^{\beta+\alpha}$.
Proof: By Definition
$g_H^{\alpha+\beta}(v) = g_H^{\alpha+\beta}[h(t)](v) =$
$\int_{-\infty}^{\infty} K_H^{\alpha+\beta}(t\,,v)h(t)dt$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} K_H^{\alpha}(t\,,v')K_H^{\beta}(v',v)dv'\right]h(t)dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} K_H^{\alpha}(t\,,v')h(t)dt\right]K_H^{\beta}(v',v)dv'$$

$$= \int_{-\infty}^{\infty}[g_H^{\alpha}[h(t)](v')]K_H^{\beta}(v',v)dv'$$

$= g_H^{\beta}[g_H^{\alpha}[h(t)](v')](v)$
$= g_H^{\beta}\circ g_H^{\alpha}[h(t)](v)$
$= g_H^{\beta}\circ g_H^{\alpha}(v)$.
That is $g_H^{\alpha+\beta} = g_H^{\alpha}\circ g_H^{\beta}(v) = g_H^{\beta}\circ g_H^{\alpha}(v)$.

Property. 7: Associativity.

$$g_H^{\alpha}\circ\left(g_H^{\beta}\circ g_H^{\gamma}\right) = \left(g_H^{\alpha}\circ g_H^{\beta}\right)\circ g_H^{\gamma}.$$

Property. 8: Shifting. The shifting property for the FRHT is written as

$$g_H^{\alpha}[h(t+b)](v) = \left\{\left(\frac{1+\cos\psi}{2}\right)g_H^{\alpha}(v+b\cos\psi) + \frac{i}{2}\sin\psi\ g_H^{\alpha}(-v-b\cos\psi)\right\} e^{\frac{1}{2}\,i\,b^2\,\sin\psi\cos\psi+i\,v\,b\,\sin\psi}$$
$$+\left\{-\frac{i}{2}\sin\psi\ g_H^{\alpha}(-v+b\cos\psi) + \left(\frac{1-\cos\psi}{2}\right)g_H^{\alpha}(v-b\cos\psi)\right\} e^{\frac{1}{2}\,i\,b^2\sin\psi\cos\psi-i\,v\,b\,\sin\psi},$$

where $g_H^{\alpha}(v)$ is the FRHT of $h(t)$.

The proof is obtained by relation between FRFT and FRHT, vice versa and shifting property of FRFT see in[2].

Examples

Example. 1: If $h(t) = 1$, then the FRHT of $h(t)$ with parameter $\alpha \in \mathbb{R}$ is equal to $\sqrt{1 + i\tan\psi}\; e^{-i\frac{1}{2}\, v^2 \tan\psi}$ if $\psi - \frac{\pi}{2}$ is not multiple of π.

Example. 2: If $h(t) = e^{\frac{-t^2}{2}}$, then the FRHT of $h(t)$ with parameter $\alpha \in \mathbb{R}$ is equal to $e^{\frac{-v^2}{2}}$.

Example. 3: The aperture function $p_\gamma(x) = 1$ if $|x| < \gamma$ and 0 otherwise then FRHT of $p_\gamma(x)$ is

$$\left[2\sqrt{\frac{1-i\cot\psi}{2\pi}}\,\sin\gamma\,\exp\left\{i\frac{1}{2}(\gamma^2 + v^2)\cot\gamma\right\}\left\{\frac{1}{v^2-\gamma^2\cos\gamma^2}\right\}[i\gamma\cos\gamma\cos(\gamma v\ \ cosec\,\gamma) + v\sin(\gamma v\ \ cosec\,\gamma)]\right].$$

For the solutions of Examples 1, 2, 3 use Result 3 and the value of FRFT of $h(t) = 1$ and $h(t) = e^{\frac{-t^2}{2}}$ by[2].

Conclusion:

In this paper inversion formula for fractional Hartley transform is obtained. Also proved some properties of fractional Hartley transform and its kernel such as symmetry, conjugate of kernel and linearity, shifting of transform and some examples.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at Shri Chhatrapati Shivaji College, Omerga, Osmanabad, Maharashtra,India.

References:

1. Alieva T, Lopez V, Agullo-Lopez F, Almeida LB. The fractional Fourier transform in optical propagation problems. J Mod Opt. 1994; 41(5): 1037-1044. http://dx.doi.org/10.1080/09500349414550971.
2. Gaikwad V, Chaudhary MS. n-dimensional fractional Fourier transform. Bull Cal Math Soc. 2016; 108(5): 375-390. https://www.researchgate.net/publication/368309310_n-dimensional_fractional_Fourier_transform.
3. Yu SS, Zhou NR, Gong LH, Nie Z. Optical image encryption algorithm based on phase-truncated short-time fractional Fourier transform and hyper-chaotic system. Opt Lasers Eng. 2020; 124: 105816. http://dx.doi.org/10.1016/j.optlaseng.2019.105816.
4. Pei SC, Tseng CC, Yeh MH, Ding JJ. A new definition of continuous fractional Hartley transform. International Conference on Acoustics, speech, and signal processing, IEEE.: 1998;3:1485-1488. http://ntur.lib.ntu.edu.tw/bitstream/246246/2007041910021806/1/00681730.pdf.
5. Roopkumar R, Ganesan C. Fractional Hartley transform on G-Boehmian space. Bol Soc paran Mat.2022;40:1-11. http://dx.doi.org/10.5269/bspm.43828.
6. Yadav AK., Singh P, Singh K. Cryptosystem based on devil's vortex Fresnel lens in the fractional Hartley domain.:J.Opt.2018;47(2):208–219. https://link.springer.com/article/10.1007/s12596-017-0435-9.
7. Singh P, Yadav AK., Singh K, Saini I. Asymmetric watermarking scheme in fractional Hartley domain using modified equal modulus decomposition. J Optoelectron Adv Mater. 2019; 21(7–8): 484–491. https://joam.inoe.ro/articles/asymmetric-watermarking-scheme-in-fractional-hartley-domain-using-modified-equal-modulus-decomposition/fulltext.
8. Yadav AK, Singh P, Saini I, Singh K. Asymmetric encryption algorithm for colour images based on fractional Hartley transform. J Mod Opt. 2019; 66(6):629–642. https://doi.org/10.1080/09500340.2018.1559951.
9. Ye HS, Zhou NR, Gong LH. Multi-image compression-encryption scheme based on quaternion discrete fractional Hartley transform and improved pixel adaptive diffusion. Signal Process. 2020; 175: 107652. http://dx.doi.org/10.1016/j.sigpro.2020.107652.
10. Kumar J, Singh P, Yadav AK. Asymmetric double-image encryption using twin decomposition in fractional Hartley domain. Opt Appl. 2022; 52(1): 21-35. http://dx.doi.org/10.37190/oa220102.
11. Rimani R, Said NH, Ali-Pacha A, Ramos JAL. An efficient image encryption using a dynamic nonlinear and secret diffusion scheme. Baghdad Sci J. 2021; 18(3):628-639. http://dx.doi.org/10.21123/bsj.2021.18.3.0628.
12. Hamza YA, Tewfiq NE, Ahmed MQ. An enhanced approach of image steganographic using discrete Shearlet transform and secret sharing. Baghdad Sci J. 2022;19(1):197-207. http://dx.doi.org/10.21123/bsj.2022.19.1.0197.

تحويل هارتلي الكسري ومعكوسه

فاسانت جايكواد

قسم الرياضيات، ش .كلية شاترباتي شيفاجي، أوميرجا، عثمان أباد ، ماهاراشترا ، رقم التعريف الشخصي 413606 ، الهند.

الخلاصة:
يُعمم تحويل هارتلي على تحويل هارتلي الجزئي (FRHT) الذي يعطي استخدامات مختلفة في مجالات مختلفة لتشفير الصور. لسوء الحظ، فإن الأدبيات المتوفرة حول تحويل هارتلي الكسري غير قادرة على توفير نظرية الانعكاس الخاصة بها. لذلك لا يمكن استرداد الدالة الأصلية بشكل مباشر، مما يقيد تطبيقاتها. تهدف هذه الورقة إلى اقتراح نظرية الانعكاس لتحويل هارتلي الجزئي للتغلب على هذا العيب. علاوة على ذلك، تمت مناقشة بعض خصائص تحويل هارتلي الجزئي في هذا البحث.

الكلمات المفتاحية :تحويل فورييه، تحويل فورييه الجزئي، تحويل هارتلي الكسري، تحويل هارتلي، تحويل معكوس.