Open Access **Baghdad Science Journal** **P-ISSN: 2078-8665**
2023, 20(1 Special Issue) ICAAM: 400-408 **E-ISSN: 2411-7986**

DOI: https://dx.doi.org/10.21123/bsj.2023.8411

Quotient on some Generalizations of topological group

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ICAAM= International Conference on Analysis and Applied Mathematics 2022.

Received 20/1/2023, Revised 19/2/2023, Accepted 20/2/2023, Published 6/3/2023

Abstract:
In this paper, some generalizations of the topological group namely α-topological group, b-topological group, and β-topological group were defined with illustrative examples. In addition, the grill topological group with respect to a grill was defined. Later, the quotient on generalizations of the topological group in the particular p-topological group was deliberated. Moreover, the model of the robotic system which relays on the quotient of p-topological group was discussed.

Keywords: b-Topological group, Generalized open, Grill topological group, Quotient space, Topology.

Introduction:
In topology, various generalizations of open sets are available namely, regular open, α-open, semi-open, pre-open, b-open, and β-open. Based on the continuity of those generalized open sets, different kinds of topological groups namely, S-topological group by Bosan [1], s-topological group by E. Bohn [2], Almost topological group by Madhu Ram [3] and p-topological group by the authors ourself [4] were defined recently. By cogitating topology and ideal on a group, P. Gnanachandra defined β-Ideal topological group [5] and distinct generalizations of ideal topological groups were discussed in [6].

The pair (K, Υ) connotes a group K and a topology Υ on K with no separation axioms assumed throughout this work. A subset S of K is regular open (respectively, α-open, semi-open, pre-open, b-open, β-open) if $M = int(cl(M))$[7], (respectively, $M \subseteq int\left(cl(int(M))\right)$[8], $M \subseteq cl(int(M))$[9], $M \subseteq int(cl(M))$[10], $M \subseteq int(cl(M)) \cup cl(int(M))$[11], $M \subseteq cl\left(int(cl(M))\right)$[12]). For a subset M of K, $int(M)$, $\alpha int(M)$, $sint(M)$, $pint(M)$, $bint(M)$ and $\beta int(M)$ denote interior, α-interior, semi-interior, pre-interior, b-interior, and β-interior of M in K respectively. In addition, $cl(M)$, $\alpha cl(M)$, $scl(M)$, $pcl(M)$, $bcl(M)$, and $\beta cl(M)$ denote closure, α-closure, semi-closure, pre-closure, b-closure, and β-closure of M in K respectively. The set $T \subseteq \mathrm{K}$ is symmetric if $T = T^{-1}$, where $T^{-1} = \{k^{-1}: k \in T\}$. The power set of K is denoted by $\mathrm{P(K)}$. An ideal I on K [13] is a subset of P(K) which satisfies (i) $S \in I$ and $T \subseteq S$ implies $T \in I$, (ii) $S, T \in I$ implies $S \cup T \in I$. In an ideal topological space $(\mathrm{K}, \Upsilon, I)$, for a subset T of K, local function [14] with respect to ideal I and topology Υ is given by $T^*(I, \Upsilon) = \{g \in K : E \cap T \notin I\}$. For convention, T^* denotes $T^*(I, \Upsilon)$. In addition, $T \subseteq \mathrm{K}$ is ideal-open if $T \subseteq int(T^*)$[13]. A non-empty collection G of P(K) is a grill [15] if (i) $\emptyset \notin G$, (ii) $S \in G$ and $S \subseteq T$ implies $T \in G$ (iii) $S, T \subseteq K$ and $S \cup T \in G$ implies $S \in G$ or $T \in G$. In a grill topological space (K, Υ, G), an operator $\varphi: \mathrm{P(K)} \to \mathrm{P(K)}$ is defined by $\varphi(T) = \{g \in K: E \cap T \in G\}$ for each neighborhood E of g. Moreover, T is grill-open [15] if $T \subseteq int(\varphi(T))$. Left translation and right translation are meant by $l_a(x) = ax$ and $h_a(x) = xa$ respectively. The collection of all open (respectively, α-open, semi-open, pre-open, b-open, β-open, ideal-open, grill-open, closed, α-closed, semi-closed, pre-closed, b-closed, closed, ideal-closed, grill-closed) sets in a topological space K is denoted by $O(\mathrm{K})$, $\alpha O(\mathrm{K})$, $PO(K)$, $SO(K)$,

$bO(K)$, $\beta O(K)$, $IO(K)$, $GO(K)$, $C(K)$, $\alpha C(K)$, $SC(K)$, $PC(K)$, $bC(K)$, $\beta C(K)$, $IC(K)$, GC(K).

Preliminaries:

In this section, Preliminary results which will be required in the next section were discussed.

Definition. 1: Let S, T be topological spaces. The map $\eta: S \to T$ is pre-continuous [10] if $\eta^{-1}(E) \in PO(S)$ for each $E \in O(T)$ and η is open (respectively, pre-open [10]) if $\eta(F) \in O(T)$ (respectively, $\eta(F) \in PO(T)$) for each $F \in O(S)$. In addition, η is pre-quotient with the provision $E \in O(T)$ if and only if $\eta^{-1}(E) \in PO(S)$. The map η is p-open [16] if $\eta(F) \in PO(T)$, for every $T \in PO(S)$.

Definition. 2: A bijective map $\eta: S \to T$ is homeomorphism [17] (respectively, p-homeomorphism [18]) if η is bi-continuous (respectively, pre-continuous and pre-open). In addition, S is a homogeneous space (respectively, p-homogeneous space [4]) if for any $m, n \in S, \exists$ a homeomorphism (respectively, p-homeomorphism) η such that $\eta(m) = n$.

Lemma. 1:[19] Let M, N be subsets of K such that $N \subseteq M$ and $M \in PO(K)$. Then, $N \in PO(K)$ if and only if $N \in PO(M)$.

Definition. 3:[4] A pair (K, Υ) is p-topological group, for each $c, d \in K$:
(i) for each $E \in O(K)$ of cd, $\exists\, C, D \in PO(K)$ of c and d such that $CD \subseteq E$,
(ii) for each $F \in O(K)$ of $k^{-1}\, \exists\, G \in PO(K)$ of k such that $G^{-1} \subseteq F$.

Definition. 4:[2] (K, Y) is s-topological group if for each $c, d \in K$ and each $E \in O(K)$ of $cd^{-1} \exists\, C, D \in SO(K)$ of c and d such that $CD^{-1} \subseteq E$.

Definition. 5.[5] A 3-tuple (K, Υ, I) is an ideal topological group if for each $c, d \in K$ and each $E \in O(K)$ of $cd^{-1}\ \exists\, C, D \in IO(K)$ of c and d such that $CD^{-1} \subseteq E$.

Generalizations of topological groups

In this section, some generalized notions of topological groups were defined, and discuss some results.

Definition. 6: A pair (K, Υ) is b-topological group (respectively, α-topological group, β-topological group) if:
(i) for each $L \in O(K)$ of mn, $\exists\, M \in bO(K)$ (respectively, $\alpha O(K), \beta O(K)$) of m and $N \in bO(K)$ (respectively, $\alpha O(K), \beta O(K)$) of n such that $MN \subseteq L$,
(ii) for each $S \in O(K)$ of $t^{-1}, \exists\, T \in bO(K)$ (respectively, $\alpha O(K), \beta O(K)$) of t such that $T^{-1} \subseteq T$.

Definition. 7: A tuple (K, Υ, G) is a grill topological group if:
(i) for each $L \in O(K)$ of $mn, \exists\, M, N \in GO(K)$ of m and n such that $MN \subseteq L$,
(ii) for each $S \in O(K)$ of $t^{-1}, \exists\, T \in GO(K)$ of t such that $T^{-1} \subseteq T$.

By the relations stated in [20, 21], It is clear that α-open $\Rightarrow$ semi-open $\Rightarrow$ b-open $\Rightarrow$ β-open, α-open $\Rightarrow$ pre-open $\Rightarrow$ b-open $\Rightarrow$ β-open and every ideal can associate a grill in which the local operators coincide. Then, the following implications can be stated:
(i) α-topological group $\Rightarrow$ s-topological group $\Rightarrow$ b-topological group $\Rightarrow$ β-topological group.
(ii) α-topological group $\Rightarrow$ p-topological group $\Rightarrow$ b-topological group $\Rightarrow$ β-topological group.
(iii) Any Ideal topological group can associate with a Grill topological group and vice versa.

But, the reverse implication of (i) and (ii) is not true in general, by the following examples.

Example. 1: The set of real numbers $\mathbb{R}$ conjoint with lower limit topology under usual addition is b-topological group which is not a p-topological (respectively, s-topological) group.

Example. 2: Klein four group k_4 with $\Upsilon = \{\emptyset, \{e, a, b\}, k_4\}$ is p-topological group which is not an α-topological (respectively, s-topological) group.

Example. 3: $(\mathbb{R}, +)$ with topology generated by the basis $\{(m, n), (k, l\,] : k < l < 0\}$ is s- topological group which is not an α-topological (respectively, topological) group.

Example. 4: Let K be any group with order greater than 2 and $a \in K$, binded with a topology $\Upsilon = \{\emptyset, K \setminus \{a\}, K\}$ and an ideal $I = \{\emptyset, \{a\}\}$. Then, $\{a\}^* = \emptyset$. For $T \subseteq K$ other than $\{a\}$, $T^* = K$. Thus, $IO(K) = P(K) \setminus \{a\}$. By Definition 5, (K, Υ, I) is an ideal topological group. Let G be a grill whose elements do not belong to I. Then (K, Υ, G) is a grill topological group.

From the above examples, observe that generalizations that initiate with the closure operator on a set exist more rather than the generalizations which initiate with the interior operator. During the process of generalization of the topological group, the basic factual to check is the translation of an open set and a closed set and the possibilities to attain a generalized open set by translating another generalized open set. For convenience, the following Proposition is proven for a b-topological group which can be similarly proven for α-topological group and β-topological group.

Proposition. 1: Suppose K is a b-topological group and $t \in K$ is arbitrary. Then, the following results hold:

(i) if $M \in O(K)$, then $tM, Mt \in bO(K)$,

(ii) if $S \in C(K)$, then $tS, St \in bC(K)$.

Proof:

(i) Let $a \in t\,M$, then $a = tm$ for some $m \in M$. Now, $m = t^{-1}a$ and by Definition 6, $\exists\, E, F \in bO(K)$ of t^{-1}, a respectively such that $EF \subseteq M$ which implies $a \in F \subseteq tM$. Hence, $tM \in bO(K)$ and in a similar way, $Mt \in bO(K)$ can be proved.

(ii) Let $a \in bcl\,(tS)$ and $E \in O(K)$ of s with $s = t^{-1}a$. Then by Definition 6, $\exists\, C, DbO(K)$ of t^{-1}, a respectively, with $CD \subseteq E$. Since $a \in bcl\,(t\,S)$, it implies that $D \cap tS \neq \emptyset$. Let $d \in D \cap tS$, then $t^{-1}d \in S \cap CD \subseteq S \cap E$. Thus, $S \cap E \neq \emptyset$. Hence, s is a limit point of S and since $S \in C(K), s \in S$. Now, $a = ts$ and so, $a \in tS$. By the above argument, $bcl(tS) \subseteq t\,S$ and since $tS \subseteq bcl(tS)$ is trivial, thus $tS = bcl\,(tS)$. Hence, $tS \in bC(K)$. The proof of $St \in bC(K)$ can be obtained similarly.

Generalized quotient structures

Quotient on a group K imparts an equivalence relation $\sim$ on K. To talk about the quotient on the topological group, that invariant subgroup should topologically strengthen itself and quotients on generalizations of topological groups (respectively, ideal topological groups) become topological groups (respectively, ideal topological groups).

Let K be a p-topological group and M be an invariant subgroup of K. Consider the equivalence relation on K defined by $s \sim t \Leftrightarrow sM = tM$, where sM, tM are left cosets of M by $s, t \in K$. Since M is invariant, each left coset is also a right coset of M in K. The multiplication of cosets in K/M is defined by $(sM)(tM) = stM$, for all $s, t \in K$. Hence, K/M is a quotient group of K with respect to M. The canonical projection $\rho: K \to K/M$ given by $\rho(a) = aM$, induces quotient topology on K/M, the finest topology for which ρ is pre-continuous. Thus, $T \subseteq K/M \in O(K/M)$ if and only if $\rho^{-1}(T) \in PO(K)$. Moreover, ρ is pre-quotient and topology on K/M is discrete if $M \in O(K)$ or the index of M is finite.

Theorem. 1: Suppose K is a p-topological group and M is an invariant closed subgroup of K. Then the following results hold:

(i) the quotient map ρ is p-open, open and pre-continuous,

(ii) $\{\rho(sG) : G \in O(K), e \in G\}$ is a local base of K/M at $sM \in K/M$,

(iii) the topology on K/M is a T_1-space,

(iv) the space K/M is homogeneous, ηa is a homeomorphism of K/M, respectively, and $\rho \circ \lambda_a = \eta_a \circ \rho$ where λa is the left translation of K by a and η_a is the left translation of K/M by a for each $a \in K$.

Proof:

(i) Since ρ is a pre-quotient map, thus pre-continuity is obvious. Let $H \in PO(K)$ be arbitrary and $E \in O(K)$ of identity e. Given that K is a p-topological group, and so $Em \in PO(K)$, for all $m \in MH$. Since any union of pre-open sets is pre-open, then EMH is pre-open and it is the minimal union of all co sets which covers H. Now, the surjectivity of ρ, implies that $\rho^{-1}(\rho(EMH)) = EMH$ and $\rho(H) = \rho(EMH)$. Since ρ is pre-quotient, thus $\rho(H) \in O(K/M)$. Since, every open set is pre-open, hence the image of a pre-open set is pre-open and so ρ is a p-open mapping. Also, ρ maps open sets to open sets, and hence ρ is open.

(ii) Let $E \in O(K/M)$ of sM in K/M be arbitrary and $F = \rho^{-1}(E)$. Since ρ is pre-quotient and $E \in O(K/M)$ it implies that $F \in PO(K)$ with $sM \subseteq F$. Then, $\exists\, G \in O(K)$ of e and $H \in PO(K)$ of s such that $sG \cap H \subseteq F$. Now, $\rho(sG \cap H) = \rho(sG) \subseteq \rho(F) \subseteq E$ and so $\rho^{-1}(\rho(sG)) \subseteq F$. Since $sGM = \rho^{-1}(\rho(sG))$, thus $\rho(sGM) \subseteq E$. Hence, $\{\rho(sG)\}$ is a local base of $sM \in K/M$.

(iii) Since $M \in C(K)$ and translations are pre-continuous in K, hence $sM \in PC(K)$. Now, $Mc \cap (sM)c = (M \cup sM)c \in PO(K)$, since it is the intersection of an open set and a pre-open set. Also, ρ is pre-quotient and so $\rho((M \cup sM)c) \in O(K/M)$. Since $sM \in K/M$ is arbitrary, every finite set in K/M is closed and so K/M is a T_1-space.

(iv) Define $\eta_a : K/M \to K/M$ by $\eta_a(sM) = asM$. Since K/M is a group, thus $asM \in K/M$, for all $a \in K, sM \in K/M$. Then, η_a is a bijection of K/M onto itself. Consider $sM \in K/M$ and $E \in O(K)$ of e, then by (ii), $\rho(sEM)$ is a base neighborhood of sM in K/M. Similarly, $\rho(asEM)$ is a base neighborhood of asM in K/M. Since $\eta_a(\rho(sE)) = \rho(asEM)$, thus η_a is open and continuous. Hence, η_a is a homeomorphism. Also, $\rho \circ \lambda_a = \eta_a \circ \rho$ because $\eta_a(sM) = asM$. Now, for any given sM and tM in K/M, consider $a = ts^{-1}$. Then, $\eta_a(sM) = tM$. Hence, the quotient space K/M is homogeneous.

Theorem. 2: Suppose K is a p-topological group, M is a closed subgroup of K and ρ is the natural canonical mapping of K onto K/M, then the following results hold:

(i) if $S, T \in O(K)$ of identity e in K, such that $T^{-1}T \subseteq S$, then $\overline{\rho(T)} \subseteq \rho(S)$,

(ii) the quotient space K/M is regular.

Proof:

i. Let $s \in K$ be arbitrary such that $\overline{\rho(s)} \in \rho(T)$. Since $T \in O(K)$ of e in K, thus $Ts \in PO(K)$ of s. The map ρ is pre-quotient and so, $\rho(Ts) \in O(K/M)$ of $\rho(s)$. Therefore, $\rho(Ts) \cap \rho(T) \neq \emptyset$ and $\rho(ms) = \rho(n)$ for some $m, n \in T$. Now, $ms = nt$, for some $t \in M$. Since $m^{-1}n \in T^{-1}T \subseteq S$, thus $s = (m^{-1}n)t \in \underline{SM}$. Thus, $\rho(s) \in \rho(SM) = \rho(S)$ and hence, $\overline{\rho(T)} \subseteq \rho(S)$.

ii. Let ρ be the quotient mapping of K onto $K/M, E \in O(K/M)$ of $\rho(e) = M$ in K/M be arbitrary, where e is an identity of K. Since K/M is a topological group and $E \in O(K/M)$ then $\exists\, F \in O(K/M)$ such that $F^{-1}F \subseteq E$. Again K/M is T_1-space, and so $M = \rho(e)$ is closed in K/M. Let η be the quotient of M on K/M. By (i), $\overline{\eta(F)} \subseteq \eta(E)$. Since $\eta(B) = B$ for all $B \subseteq K/M$, thus, $\bar{F} \subseteq E$. Hence, K/M attains regularity at $\rho(e)$. Since K/M is homogeneous, thus K/M is regular.

Proposition. 2: Let $\chi: K \to M$ be a homomorphism of p-topological groups which is pre-continuous with, for each $F \in PO(K)$ of identity eK in K, the range $\chi(F)$ contains a non-empty open set in M. Then, the homomorphism χ is pre-open.

Proof:

Let $E \in O(K)$ of eK be arbitrary. Since K is a p-topological group, it is possible to find $F \in PO(K)$ such that $F^{-1}F \subseteq E$. By hypothesis, $\chi(F)$ contains $G \in O(M)$, which is non-empty. Then, $G^{-1}G \in PO(M)$ of e_M. Since χ is a homomorphism, thus $G^{-1}G \subseteq \chi(F)^{-1}\chi(F) = \chi(F^{-1})\chi(F) = \chi(F^{-1}F) \subseteq \chi(E)$. So, $e_M \in pint(\chi(E))$ for each $EO(K)$ of e_K. Now, choose $n \in \chi(E)$, where $E \in O(K)$ are arbitrary. Then, find $m \in E$ with $\chi(m) = n$ and $F \in PO(K)$ of e_K such that $mF \subseteq E$. Let $G \in O(M)$ of e_M with $G \subseteq \chi(F)$. Then, the set nG contains n and it is pre-open in M. Hence, $nG \subseteq \chi(mF) \subseteq \chi(E)$. It implies that $\chi(E) \in PO(M)$ and hence, the homomorphism χ is pre-open.

Proposition. 3: Let K and M be p-topological groups with $\chi: K \to M$ an onto homomorphism which is continuous such that, for some non-empty $E \in PO(K)$, the set $\chi(E) \in O(M)$ and the restriction of χ to E is a pre-quotient mapping of E onto $\chi(E)$. Then, the homomorphism χ is pre-open.

Proof:

Let $G \in O(K)$ of n. Let $m \in E$ be arbitrary and l be the left translation of K by mn^{-1}. Then, l is a p-homeomorphism of K onto itself such that $l(n) = m$ and by Lemma 1, $F = E \cap l(G) \in PO(E)$ of m. Since the restriction of χ is pre-quotient on E, then $\chi(F) \in O(M)$. Consider, the left translation h of M by the inverse to $\chi(mn^{-1})$. Since χ is a homomorphism, so $\left(\chi(mn^{-1})\right)^{-1} = \chi((mn^{-1})^{-1}) = \chi(nm^{-1})$. It is evident that, $h \circ \chi \circ l = \chi$ and $h(\chi(l(G))) = \chi(G)$. However, h is a p-homeomorphism of M onto itself. Since $\chi(F) \in O(M)$, it follows that $h(\chi(F)) \in PO(M)$, i.e. $\chi(G)$ contains $h(\chi(F)) \in PO(M)$ of $\chi(n)$ in M. Thus, $\chi(G) \in PO(M)$ of $\chi(n)$, and hence, the map χ is pre-open.

Theorem. 3: Let K be a p-topological group, M be a closed subgroup of K, and $\rho: K \to K/M$ be the cannonical mapping. If N is a pre-dense subgroup of K, then the restriction ρN is an open mapping of N onto $\rho(N)$ if and only if $N \cap M$ is pre-dense in M.

Proof:

Suppose that the map ρ_N from N onto $\rho(N)$ is open. Let $E \in O(K)$ of identity e_K in K. Claim that $M \subseteq E(N \cap M)$. Indeed, take a symmetric $F \in PO(K)$ of e_K such that $F^2 \subseteq E$. Then, $\rho_N(F \cap N) = \rho(F \cap N) \in O(\rho(N))$ of $\rho(e)$, so there exists $D \in O(K/M)$ such that $\rho(e) \in D \cap \rho(N) \subseteq \rho(F \cap N)$. This implies that $(\rho^{-1}(D) \cap N)M \subseteq (F \cap N)M$. Let $s \in M$ be arbitrary. Since N is pre-dense in K, $\exists\, t \in N$ such that $ts^{-1} \in F \cap \rho^{-1}(D)$. Then, $ts^{-1} \in \rho^{-1}(D) \cap NM$. So, pick $a \in F \cap N$ and $b \in M$ such that $ts^{-1} = ab$. It follows that $a^{-1}t = bs \in N \cap M$ and $b^{-1} = (st^{-1})a \in F^{-1}F \subseteq E$. So, $s = b^{-1}(a^{-1}t) \in E(N \cap M)$ and therefore, $M \subseteq E(N \cap M)$. Since the later inclusion holds for $E \in O(K)$ of e_K, the set $N \cap M$ must be pre-dense in M. Indeed, find an element $h \in M$ and a symmetric $EE \in O(K)$ of e_K such that $Eh \cap (N \cap M) = \emptyset$, whence $h \in M \setminus E(N \cap M) \neq \emptyset$, a contradiction.

Conversely, suppose that $N \cap M$ is pre-dense in M. Consider $E \in O(N)$ to be arbitrary. Then $E = F \cap N$, for some $F \in O(K)$. Since ρ is open, the set $D = \rho(F) \cap \rho(N) \in O(\rho(N))$, and claim that $\rho_N(E) = D$. Indeed, if $t \in D$, consider a point $s \in N$ with $\rho(s) = t$. Then, $sM \cap F = \rho^{-1}(t) \cap F \neq \emptyset$. Since $N \cap M$ is pre-dense in M, thus $s(N \cap M) = N \cap sM$ is pre-dense in xM. Hence, $(N \cap sM) \cap F \neq \emptyset$. Consider a point $a \in N \cap sM \cap F$. Then, $a \in E$ and $\rho_N(a) = \rho(a) = \rho(s) = t$, which implies that $\rho_N(E) = D$. Therefore, the mapping $\rho_N : N \to \rho(N)$ is open.

Theorem. 4: Let S, R be p-topological groups where R is a T_1-space and $\chi: S \to R$ be an onto homomorphism which is continuous and pre-quotient. Then, the kernel $M = \chi^{-1}(e_R)$ is a closed invariant subgroup of S, where e_R is identity in R and $\chi^{-1}(t)$, $t \in R$ coincide with the co sets of M in S. The map $\varphi: S/M \to R$ which assigns sM to $\chi(s) \in R$ is both isomorphism and homeomorphism.

Proof:

Given that R is T_1-space and so, e_R is closed. Also, given that χ is a continuous homomorphism, then $M = \chi^{-1}(e_R)$ is a closed invariant subgroup of S. Since χ is an onto homomorphism, thus χ^{-1} exists and it is also a homomorphism. Now, $\{t\} \in C(R)$ and pre-quotient of χ, implies that $\chi^{-1}(t) \in PC(S)$ and coincides with cosets of M in S. Let ρ be the quotient map of S onto S/M. By the definition of φ, $\chi = \varphi \circ \rho$ and so, φ is a homomorphism. Since the maps χ, ρ are onto and $\chi = \varphi \circ \rho$, φ is onto. Moreover, if $s \in S$ with $\varphi(sM) = e_R$, then $\chi(s) = \varphi(\rho(s)) = e_R$. Hence, $s \in M$ and $sM = M$. Thus, φ is one-one and hence, φ is an isomorphism. Since ρ is open, χ is continuous and $\varphi^{-1}(E) = \rho(\chi^{-1}(E))$ is open for all $E \in O(R)$ and hence, φ is continuous. Now, if $E \in O(S/M)$, then the image $\varphi(E) = \chi(\rho^{-1}(E)) \in O(R)$. Thus, φ is an open continuous bijective map on $O(S/M)$ and hence, it is a homeomorphism.

Theorem. 5: Let S, R be p-topological groups with R is a T_1-space and $\chi: S \to R$ be an onto homomorphism which is pre-quotient and continuous. Let M be a closed invariant subgroup of R, $N = \chi^{-1}(M)$ and $P = \chi^{-1}(e_R)$, where e_R is the identity of R. Then, the topological groups S/N, R/M and $(S/P)/(N/P)$ are isomorphic and homeomorphic.

Proof:

Let φ be the canonical homomorphism of R onto R/M. By Theorem 1 (i), φ is open, so the composition $\eta = \varphi \circ \chi$ is a pre-quotient continuous homomorphism of S onto R/M with kernel $N = \chi^{-1}(M)$. By Theorem 4, the quotient group S/N is isomorphic and homeomorphic to R/M. In addition, P is a closed invariant subgroup of S. Also, observe that the mapping μ assigning to every coset sP of P in S to the element $\chi(s) \in R$ is an isomorphism and homeomorphism of S/P onto R, by Theorem 4, and $\mu(N/P) = M$. Since μ is onto, continuous and M is closed, it is clear that $\mu^{-1}(M) = N/P$ is closed in S/P. Let $\omega: S/P \to (S/P)/(N/P)$ be the quotient homomorphism. Define $\sigma: (S/P)/(N/P) \to R/M$ by $\sigma(t\, N/P) = r\, M$, where $t \in S/P$ and $r = \mu(t)$. By this, $\varphi \circ \mu = \sigma \circ \omega$. Since μ, ω, φ are onto, open, continuous homomorphism and so σ. Let $t\, N/P$ be arbitrary in $(S/P)/(N/P)$ and $r = \mu(t)$. If $\sigma(t\, N/P) = M$, then $\varphi(r) = M$ so that $r \in M$ and $t \in N/P$. Thus, the kernel of σ is trivial and σ is one-one. Thus, σ is an isomorphism and so, σ is an isomorphism and homeomorphism. Hence, the group $(S/P)/(N/P)$ is isomorphic and homeomorphic to R/M.

Theorem. 6: Let S be a p-topological group, M be a closed invariant subgroup of S, and P be any p-topological subgroup of S. Then, the quotient group PM/M is isomorphic and homeomorphic to the subgroup $\rho(P)$ of the topological group S/M, where $\rho: S \to S/M$ is the natural canonical homomorphism.

Proof:

It is trivial that, $PM = \rho^{-1}(\rho(P))$. Since the map ρ is pre-continuous and open, the restriction φ of ρ to PM is a pre-continuous open mapping of PM onto $\rho(P)$. Since ρ is a homomorphism of S onto S/M and P is a subgroup of S, thus it follows that $\rho(P)$ and PM are subgroups of the groups S/M and S, respectively, and φ is a homomorphism of PM onto $\rho(P)$. Let eS be the neutral element of S. It is evident that, $\varphi^{-1}(\varphi(e)) = \rho^{-1}(\rho(e)) = M$, i.e. the kernel of the homomorphism φ is M. Now, by Theorem 4 the topological groups PM/M and $\rho(P)$ are isomorphic and homeomorphic.

Modeling of robot actions using digital image

Consider a system of robots that performs m actions at n positions where m, n are even with the provision that: one cannot set a robot to perform all actions at a particular position and after fixing the robots one is allowed to change the positions of robots only once but not the performing action to complete the actions at other positions. Let us handle the situation mathematically.

For instance, let us consider $K = \{P_0, P_1, P_2, \ldots, P_9\}$ be the group of positions that a robot can move along, $M = \{A_0, A_1, A_2, \ldots, A_{11}\}$ be the group of actions performed by a robot. It is notable to mention that K is isomorphic to $\mathbb{Z}_{10}$ and M is isomorphic to $\mathbb{Z}_{12}$. Let $\Upsilon_K = \{\emptyset, \{P_1, P_3, P_5, P_7, P_9\}, K\}$ and $\Upsilon_M = \{\emptyset, \{A_0, A_1, A_3, A_4, A_6, A_7, A_9, A_{10}\}, M\}$ are topologies on K and M respectively. Then, the pairs (K, Υ_K) and (M, Υ_M) are p-topological groups (respectively, b-topological groups, β-topological groups). Also, the positions $S = \{P_0, P_2, P_4, P_6, P_8\}$ is a closed subgroup of (K, Υ_K) and the actions $T = \{A_2, A_5, A_8, A_{11}\}$ is a closed subgroup of (M, Υ_M). Thus, topological groups M/T and K/S with respect to those closed subgroups were obtained. Now, define the configuration function $k: M/T \to K/S$ by $k(t) = \begin{cases} S^c \text{ if } t = T \\ S \text{ otherwise} \end{cases}$, which assigns the identity coset of M/T to the union of non-identity cosets of K/S and the union of non-identity cosets of M/T to the identity coset of K/S. Then, the corresponding digital image of the configuration function is given in Fig. 1.

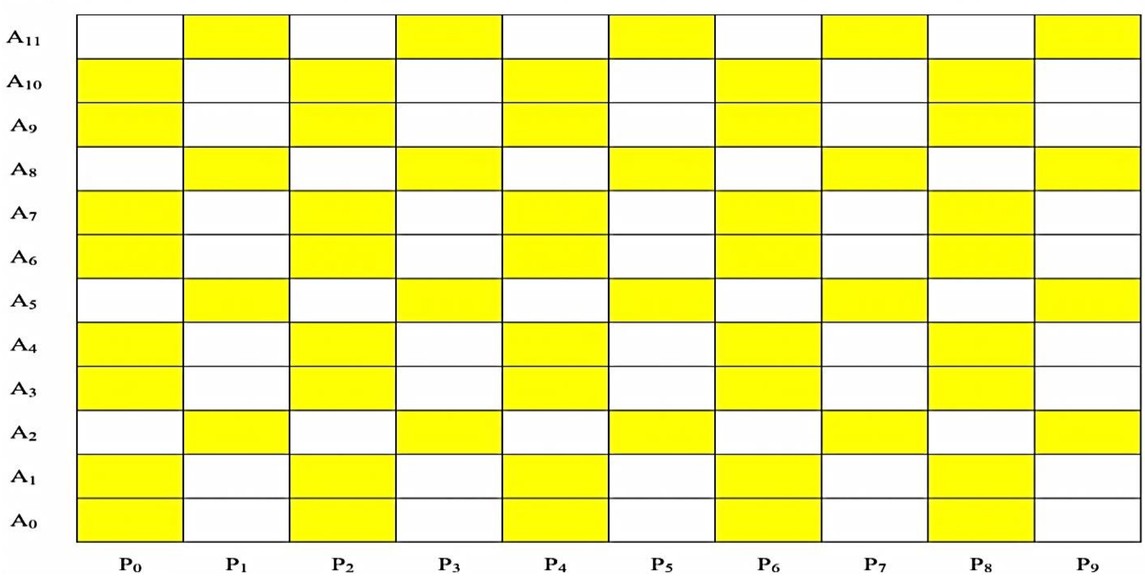

Figure 1. Digital image of the configuration function

In Fig.1, the colored boxes represent the actions that are set to do based on the configuration function. It is also clear that the digital image of the actions done by positioning the robots is equivalent to the actions yet not done. Now, by redefining the configuration function as $k^*(t) = \begin{cases} S \text{ if } t = T \\ S^c \text{ otherwise} \end{cases}$, which assigns the identity coset of M/T to the identity coset of K/S and the union of non-identity cosets of M/T to the union of identity co sets of K/S. Hence, the actions which were left undone before were completed.

The sake of futuristic research relays on the developments that evolved in the recent past decades of research. In that sense, there are eminent updates in the characterization of topological groups[22-25], distinguished forms[26-28,] and generalized forms[29, 30] of the Topological group. In addition, the structures were reformed algebraically[31-34] and topologically[35] via fuzzy metrics[36-39]. Quotient forms such as factorizable[40-42], metrizable[43,44], and separable[45-47] especially in Banach spaces[48-51] were developed recently. Bounded groups[52,53], Matrix groups[54], homotopy characteristics[55-58,] and projectivity[59] on topological groups were remarkable.

Open Questions:

Now, the following open questions were proposed for our readers.

Q1. Is it possible to obtain a pair (K, Υ) such that (K, Υ) is a β-topological group that is not b-topological?

Q2. Is it possible to obtain a pair (K, Υ) such that (K, Υ) is an α-topological group that is not a topological group?

Conclusion:

In this article, some generalized ideas of a topological group were defined and proved some characterizations of these groups. Moreover, two questions that are too open to the readers were proposed.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Ayya Nadar Janaki Ammal College.

Authors contributions:

This work was carried out in collaboration between all authors. A. Muneesh Kumar developed the research idea under the supervision of P. Gnanachandra. B. Ananda Priya contributes to proofreading and designed the model section. All authors read and approved the final manuscript.

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حاصل القسمة لبعض التعميمات لمجموعة تبولوجية

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الخلاصة:
في هذا البحث ، تم تعريف بعض التعميمات للمجموعة التبولوجية وهي المجموعة التبولوجية - α ، والمجموعة التبولوجية - ب ، والمجموعة التبولوجية - β مع أمثلة توضيحية. بالإضافة إلى ذلك ، تم تعريف المجموعة التبولوجية للشواء فيما يتعلق بالشواية. فيما بعد ، تم تداول حاصل قسمة تعميمات المجموعة التبولوجية في مجموعة تبولوجية - p معينة. علاوة على ذلك ، تمت مناقشة نموذج النظام الروبوتي الذي يعتمد على حاصل المجموعة التبولوجية – p.

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