A Fixed Point Theorem for L-Contraction in Generalized D-Metric Spaces

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Abstract

We define *L*-contraction mapping in the setting of D-metric spaces analogous to *L*-contraction mappings [1] in complete metric spaces. Also, give a definition for general D- matric spaces.And then prove the existence of fixed point for more general class of mappings in generalized D-metric spaces.

Keywords: Fixed point, L-contraction mappings, D-metric spaces.

1. Preliminaries

In [2] Dhage introduced the concept of D- metric spaces as followes

Definition 1.1 [2] Let X be a nonempty set . A function $D : X \times X \times X \to \Re^+$($\Re^+$ is the set of all non negative real numbers) is called a **D-metric spaces** on X if

i. $D(x,y,z) = 0$ if and only if $x = y = z$ (coincidence)

ii. $D(x,y,z) = D(p\{x,y,z\})$, where p is a permutation of x, y, z (symmetry)

iii. $D(x,y,z) \le D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x, y, z, a \in X$ (telrahedral inequality).

A nonempty set X , together with D-metric , is called D- metric space and denoted by (X,D). Some specific examples of D-metrics appeared in [3] and [4] .

Definition 1.2 [2] A sequence of points of a D-metric space X is said to be **D-convergent to a point** $x \in X$ if for each $\in > 0$ there exists an $n_0 \in N$ such that for all $m, n \ge n_0$, $D(x_m,x_n,x) < \in$.

Definition 1.3 [2] A sequence of points of a D-metric space X is said to be **D-cauchy sequence** if for $\in > 0$ there exists an $n_0 \in N$ such that for all $m , n , p \ge n_0$, $D(x_m,x_n,x_p) < \in$.

Definition 1.4 [5] A D-metric space X is said to be **complete** every D-cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

Definition 1.5[1] Let E be a Banach space. A subset K is called **a cone** if it is

closed, convex and $t K \subset K$ for $t \in \Re^+$ and $K \cap (-K) = 0$.

Given a cone K in E we define a partial ordinary in E by writing.

$$x < y \text{ if and only if } y - x \in K \qquad (1)$$

Definition 1.6 [1] A subset K in a Banach space E is called a **normal** if there exists $\delta > 0$ such that $0 < x < y$ implies $\|x\| \le \delta \|y\|$.

2. Main Results

Firstly we define the following

Definition 2.1 A set X is said to be a **general D-metric space** if there exists a function $D : X \times X \times X \to K$,where K is a normal cone in a Banach space, such that

i. $D(x,y,z) = 0 \in K \Leftrightarrow x = y = z$

ii. $D(x,y,z) = D(p\{x,y,z\})$ for all x,y,z in X, (p is a permutation of x,y,z)

iii. $D(x,y,z) < D(x,y,z) + D(x,a,z) + D(a,y,z)$ for all x,y,z,a inX"<denote the partial ordering induced by K

Remark

If we put $K= \Re^+$ in the definition (2.1) then the general D-metric function will be D-metric function and then the examples (1.8), (1.20) and (1.21) in [4] show that in general D-metric space : (a) D-metric does not always define a topology , (b) even D-metric define a topology , it need not be Hausdorff (therefore the limit need not be unique),

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and (c) even D-metric define a topology ,the D-metric function need not be continuous even in a single variable .Now for uniqueness limit in general D-metric spaces, we reform the concept of continuity of general D-metric function and then give a result which guarantee the uniqueness limit if exist

Definition 2.2

A general D-metric function is called in three variables if the sequence{D(x_n.y_n,z_n)} in K converges toD(x,y,z) whenever x,y,z∈X and {x_n},{y_n}and{z_n}are sequences in Xconverge to x,y and z,respectively with respect to general D-metric .

Proposition 2.3 Let(X,D)be a general D-metric space and D be continuous in three variables,then every convergent sequence in(X,D)has a unique limit.

Proof It is easy to prove this result since {D(x_n.y_n,z_n)}in K and any convergent sequence in Banach space has a unique limit .

Definition 2.4 [1] Let K be a normal cone in a Banach space E,the function ℓ:K→ $\Re^+$is a sublinear positively homogenous functional if for any u, v ∈ K then $\ell(u + v) \le \ell(u) + \ell(v)$ and $\ell(t\,u) = t\,\ell(u)$,for $t \ge 0$ such that $\ell^{-1}(0) = 0$.

Proposition 2.5 Let K be a normal cone in a Banach space E and (X, D) be a general D-metric space.If D*:X×X×X→$\Re^+$is the function defined by

D*(x,y,z) = ℓ(D(x,y,z)) (2)

Where ℓ as in the definition(2.2) then (X,D) is D-metric space.

Proof: By conditions i ,ii and iii of definition (1.1) and definition (2.2) ,one can prove $\ell(u) = \|u\|$ Naidu[4],show that the metric function D is not continuous even in one variable,therefore,through this paper the D-metric is assumed to be continuous in three variables.Note that,if D is continuous in three variables then the limit is unique in exists [4] .

Theorem 2.6 Let X be a generalized D-metric space which is complete in the metric defined by(2)with D iscontinuous in three variables and ,if T:X → X satisfies

D(Tx,Ty,Tz) ≤ L(D(x,y,z) + D(Tx,Ty,Tz))

Where L is bounded positive linear operator in E with spectral radius less than1/2, then there is a unique fixed point, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n$.

proof :

$let\, x_0 \in X,\, put\, x = T^2x_0,\, y = Tx_0,\, z = x_0$

$D(T^3x_0, T^2x_0, Tx_0) \subseteq LD(T^3x_0, T^2x_0, Tx_0) +$
$LD(T^2x_0, Tx_0, x_0) \Rightarrow (I-L)D(T^3x_0, T^2x_0, Tx_0) \subseteq LD(T^2x_0, Tx_0, x_0)$

$\sin ce\;\; r(L) \subset \frac{1}{2} \subset 1, then (1-L) is\, invertable$ [6, pp795]

$D(T^3x_0, T^2x_0, Tx_0) \subseteq (I-L)^{-1}$
$L(D(T^2x_0, Tx_0, x_0))$.

To prove that

$D(T^{n+1}x_0, T^nx_0, T^{n-1}x_0) \subseteq (I-L)^{-(n+1)} L^{n+1}$
$[D(T^2x_0, Tx_0, x_0)] \quad n \supseteq 2 \ldots 3$

Assume that (3) is true for $n = 2$ then

$D(T^{n+1}x_0, T^nx_0, T^{n-1}x_0) \subset LD(T^{n+1}x_0, T^nx_0, T^{n-1}x_0) +$
$LD(T^nx_0, T^{n-1}x_0, T^{n-2}x_0) \Rightarrow$

(I – L) D($T^{n+1}x_0$, T^nx_0 , $T^{n-1}x_0$) < L D(T^nx_0 , $T^{n-1}x_0$, $T^{n-2}x_0$)

⇒D($T^{n+1}x_0$, T^nx_0 , $T^{n-1}x_0$) < $(I – L)^{-1}$ L D(T^nx_0 , $T^{n-1}x_0$, $T^{n-2}x_0$)

< $(I – L)^{-1}$ L $(I – L)^{-n}$ L^n D(T^2x_0 , Tx_0 ,x_0) (4)

Since $L(I – L)^{-n} = (I – L)^{-n}$ L (4) will be

D($T^{n+1}x_0$, T^nx_0 , $T^{n-1}x_0$) < $(I – L)^{-(n+1)}$ L^{n+1} D(T^2x_0 , Tx_0 ,x_0) and (3) is proved.

Furthermore, by condition (3) of D-metric, we have:

D($T^{n+m+1}x_0$, $T^{n+m}x_0$, T^nx_0) <
D($T^{n+m+1}x_0$, $T^{n+m}x_0$, $T^{m+n-1}x_0$) +
D($T^{n+m+1}x_0$, $T^{m+n-1}x_0$, T^nx_0) +
D($T^{m+n-1}x_0$, $T^{n+m}x_0$, T^nx_0)
< $(I – L)^{-(m+n+1)}$ $L^{(m+n+1)}$ D(T^2x_0 , Tx_0 ,x_0) +
D($T^{m+n+1}x_0$, $T^{m+n-1}x_0$, T^nx_0) +
D($T^{m+n-1}x_0$, $T^{m+n}x_0$, T^nx_0)
< $(I – L)^{-(m+n+1)}$ L^{m+n+1} D(T^2x_0 , Tx_0 ,x_0) +

D($T^{m+n+1}x_0$, $T^{m+n-1}x_0$, $T^{m+n}x_0$) +
D($T^{m+n+1}x_0$, $T^{m+n}x_0$, T^nx_0)+
D($T^{m+n}x_0$, $T^{m+n-1}x_0$, T^nx_0) +
D($T^{m+n}x_0$, $T^{m+n-1}x_0$, T^nx_0).

⇒ 2 D($T^{m+n+1}x_0$, $T^{n+m}x_0$, T^nx_0) < $(I – L)^{-(m+n+1)}$ L^{m+n+1} D(T^2x_0 , Tx_0 ,x_0) +
$(I – L)^{-(m+n+1)}$ L^{m+n+1} D(T^2x_0 , Tx_0 ,x_0) + 2 D($T^{m+n}x_0$, $T^{m+n-1}x_0$,x_0)

$\Rightarrow D(T^{m+n-1}x_0, T^{m+n}x_0, T^n x_0) < (I-L)^{-(m+n+1)} L^{m+n+1} D(T^2x_0, Tx_0, x_0)$

Continue, we get:

$D(T^{m+n-1}x_0, T^{m+n}x_0, T^n x_0) < [(I-L)^{-(m+n+1)} L^{m+n+1} + (I-L)^{-(m+n)} L^{m+n} + \cdots + (I-L)^{-(n+2)} L^{n+2}] D(T^2x_0, Tx_0, x_0) + D(T^{n+1}x_0, T^n x_0, T^{n-1}x_0)$. $< [(I-L)^{-(m+n+1)} L^{m+n+1} + \cdots + (I-L)^{-(n+2)} L^{n+2}] D(T^2x_0, Tx_0, x_0) + (I-L)^{-(n+1)} L^{n+1} D(T^2x_0, Tx_0, x_0)$

$< (I-L)^{-(n+1)} L^{n+1}(\sum (I-L)^{-m} L^m D(T^2x_0, Tx_0, x_0)) = ((I-L)^{-1})^{n+1} x^1$

where x^1 is the unique solution of the $x=(I-L)^{-1} L\, x + D(T^2x_0, Tx_0, x_0)$.

By spectral mapping theorem [6,pp 798] $(I-L)^{-1} L$ is such that $r((I-L)^{-1} L) < 1$

Hence,

$D(T^{m+n+1}x_0, T^{m+n}x_0, T^n x_0) < ((I-L)^{-1})\, x^1$

Being K is normal we have:

$\| D(T^{m+n+1}x_0, T^{m+n}x_0, T^n x_0\| \le \delta \, \| ((I-L)^{-1} L)\, x^1 \|$ (5)

Since the right-hand side of (5) going to zero when $n \to \infty$, we obtain that $\{T^{n+1}x_0\}$ is Cauchy with respect to the metric D. Being (X,D) complete, we denote by y the limit of $\{T^{n+1}x_0\}$. The following inequalities hold:

$(T^{n+1}x_0, T^n x_0, Ty) < L\, D(T^{n+1}x_0, T^n x_0, Ty) + L\, D(T^n x_0, T^{n-1}x_0, y)$

$\Rightarrow D(T^{n+1}x_0, T^n x_0, Ty) < (I-L)^{-1} D(T^n x_0, T^{n-1}x_0, y)$ Finally using the normality of K:

$\| D(T^{n+1}x_0, T^n x_0, Ty\| \le \delta \, \| (I-L)^{-1} L \| \, \| D(T^n x_0, T^{n-1}x_0, y \|$ (6)

Letting $n \to \infty$ in (6) we obtain $D(y, y, Ty) \le 0$, mean $Ty = y$, and this complete the proof.

A consequence of theorem (2.6) one can prove the following result

Corollary (2.7) Let (X,D), ℓ and L as in theorem (2.10). If the mapping $T : X \to X$ satisfies the condition

$D(Tx, Ty, Tz) < L(D(x,y,z))$ for all $x, y, z \in X$

Then T has a unique fixed point in X.

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مبرهنة حول النقطة الصامدة لتطبيق L- الأنكماشية في فضاءات D- المترية المعممة

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الخلاصة

بأسلوب مشابه لما ورد في [1] من تعريف لتطبيق L- الأنكماشي في الفضاءات المترية الأعتيادية سنعرف تطبيق L- الأنكماشية في فضاءات D-المترية وكذلك نعطي تعريفاً لفضاءات D-المترية المعممة ثم نبرهن وجود نقطة صامدة لنمط من التطبيقات الأكثر عمومية في فضاءات D-المترية المعممة.